Math 540 Real Analysis — Comprehensive Exam — January 2018

Do five out of the six problems. Each problem is worth 20 points. Justify all claims.

Notation:
m is Lebesgue measure on $\mathbb{R}$.
n is a positive integer.

Problem 1. Let $0 < \beta < 1$ and suppose $(t_n), (a_n)$ are sequences with $0 \le t_n \le 1$, $a_n \ge 0$ and $\sum_n a_n < \infty$. Prove that

$$\sum_n \frac{a_n}{|s - t_n|^\beta} < \infty$$

for almost every $s \in [0,1]$.

Problem 2. Suppose $0 < p < 2$, and let

$$A = \int_{\mathbb{R}^n} \frac{\sin^2(|x|)}{|x|^{n+p}}\, dx$$

where $x = (x_1, \ldots, x_n)$ and $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$. Show $A < \infty$.

Problem 3. Let $p \in [1, \infty)$. Prove the normed space $L^p[0,1]$ is complete.

Problem 4.
(i) Suppose f is nonnegative and measurable on a σ-finite measure space $(X, \mathcal{A}, \mu)$. Prove

$$\int_X f(x)^p\, d\mu(x) = \int_0^\infty p t^{p-1} \mu\big(\{x : f(x) > t\}\big)\, dt, \qquad p \ge 1.$$

(ii) Show that if f is nonnegative and Lebesgue measurable on $[0,1]$ and satisfies

$$m\big(\{x : f(x) > t\}\big) \le \frac{C}{t^2}, \qquad t > 0,$$

for some positive constant C, then $\int_{[0,1]} f\, dm < \infty$.

Problem 5. Suppose $f : [a,b] \to [c,d]$ is absolutely continuous and bijective. Consider the inverse function $f^{-1} : [c,d] \to [a,b]$.
(i) Must it be true that $f^{-1} \in BV[c,d]$?
(ii) Must it be true that $m\big(f(E)\big) = 0$ whenever $E \subseteq [a,b]$ with $m(E) = 0$?

Problem 6. Suppose $f(t)$ is 2π-periodic and absolutely continuous, so that f' exists and integration by parts is valid. Assume $f, f' \in L^2[-\pi, \pi]$ with $\int_{-\pi}^{\pi} f(t)\, dt = 0$.
Prove

$$\int_{-\pi}^{\pi} |f(t)|^2\, dt \le \int_{-\pi}^{\pi} |f'(t)|^2\, dt,$$

and find all functions f for which equality holds.