Math 540 Real Analysis — Comprehensive Exam — January 2018

Do five out of the six problems. Each problem is worth 20 points. Justify all claims.

Notation:
\( m \) is Lebesgue measure on \( \mathbb{R} \).
\( n \) is a positive integer.

**Problem 1.** Let \( 0 < \beta < 1 \) and suppose \( (t_n), (a_n) \) are sequences with \( 0 \leq t_n \leq 1, \ a_n \geq 0 \) and \( \sum a_n < \infty \). Prove that
\[
\sum_n a_n |s - t_n|^{-\beta} < \infty
\]
for almost every \( s \in [0,1] \).

**Problem 2.** Suppose \( 0 < p < 2 \), and let
\[
A = \int_{\mathbb{R}^n} \frac{\sin^2(|x|)}{|x|^{n+p}} \, dx
\]
where \( x = (x_1, \ldots, x_n) \) and \( |x| = \sqrt{x_1^2 + \cdots + x_n^2} \). Show \( A < \infty \).

**Problem 3.** Let \( p \in (1, \infty) \). Prove the normed space \( L^p[0,1] \) is complete.

**Problem 4.**
(i) Suppose \( f \) is nonnegative and measurable on a \( \sigma \)-finite measure space \( (X, \mathcal{A}, \mu) \). Prove
\[
\int_X f(x)^p \, d\mu(x) = \int_0^\infty pt^{p-1} \mu(\{x : f(x) > t\}) \, dt, \quad p \geq 1.
\]
(ii) Show that if \( f \) is nonnegative and Lebesgue measurable on \([0,1]\) and satisfies
\[
m(\{x : f(x) > t\}) \leq \frac{C}{t^2}, \quad t > 0,
\]
for some positive constant \( C \), then \( \int_{[0,1]} f \, dm < \infty \).

**Problem 5.** Suppose \( f : [a, b] \to [c, d] \) is absolutely continuous and bijective. Consider the inverse function \( f^{-1} : [c, d] \to [a, b] \).
(i) Must it be true that \( f^{-1} \in BV[c, d] \)?
(ii) Must it be true that \( m(f(E)) = 0 \) whenever \( E \subseteq [a, b] \) with \( m(E) = 0 \)?

**Problem 6.** Suppose \( f(t) \) is \( 2\pi \)-periodic and absolutely continuous, so that \( f' \) exists and integration by parts is valid. Assume \( f, f' \in L^2(-\pi, \pi) \) with \( \int_{-\pi}^\pi f(t) \, dt = 0 \).
Prove
\[
\int_{-\pi}^\pi |f(t)|^2 \, dt \leq \int_{-\pi}^\pi |f'(t)|^2 \, dt,
\]
and find all functions \( f \) for which equality holds.