Math 540 Real Analysis-Comprehensive Exam-January2019

Do five out of six problems, each problem is worth 20 points. Justify all your claims.

(1) Let f be absolutely continuous on $\mathbb{R}$ and in $L_1(\mathbb{R})$. Define

$$f_h(x) = \frac{f(x+h) - f(x)}{h} .$$

Show that the L_1-limit $\lim_h f_h$ exists in $L_1([a,b])$ for all $a < b$. (Hint: approximate f').

(2) For a function $f \in L_1([0,1])$ recall that

$$\hat{f}(k) = \int_0^1 e^{-2\pi i k x} f(x) dx .$$

Show that for $f \in L_4([0,1])$ the sequence

$$c_n = \sum_{k+j=n} \hat{f}(k)\hat{f}(j)$$

converges to 0 for $n \to \pm\infty$.

(3) Give an example for the following (indicate justification)

(a) A monotone functions which is not absolutely continuous.

(b) A singular function which is not monotone.

(c) A monotone function $f : [0,1] \to [0,2]$ and subset $A \subset [0,1]$ of Lebesgue measure 0 such that $f(A)$ has Lebesgue measure 1.

(d) A continuous function which is not absolutely continuous.

(4) Let f be a positive μ-integrable function on $[0,1]$ and g positive such that e^g is integrable. Calculate (with proof)

$$\lim_{n\to\infty} \int_0^1 e^{g-nf(x)} d\mu(x) .$$

(5) Let $f \in L_p(\omega, \Sigma, \mu)$ be positive function on a probability space, i.e. $\mu(\Omega) = 1$. Show that

$$\int |f|^p d\mu = p \int_0^\infty \mu(f > t) t^{p-1} dt .$$

(6) Let $\{f_n\}$ be a sequence of real valued measurable functions on a measure space $(X, \mathcal{A}, \mu)$. Suppose that $\mu(X) < \infty$. Prove that the sequence $\{f_n\}$ converges to f a.e. if and only if $\{g_n\}$ converges to 0 in measure. Here the sequence $\{g_n\}$ is defined as

$$g_n(x) = \sup_{k\geq n} \left| f_k(x) - f(x) \right| .$$