Math 540 Comprehensive Examination
January 19, 2017

Do five out of six problems. Each problem is worth 20 points. Justify all claims.

Notation. m denotes the Lebesgue measure on $\mathbb{R}$. $C_0(\mathbb{R})$ denotes the space of continuous functions $g : \mathbb{R} \to \mathbb{R}$ which vanish at infinity, i.e., $\lim_{x\to\pm\infty} g(x) = 0$.

1. (a) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions. Assume that $f_n \to f$ a.e. and that $f_n \to g$ in measure. Prove that $f = g$ a.e.

(b) Give an example of a sequence of functions $f_n : [0,1] \to \mathbb{R}$ such that $f_n \to 0$ in measure, but the sequence $\big(f_n(x)\big)_{n\in\mathbb{N}}$ does not converge for any choice of $x \in [0,1]$.

2. Assume that $f \in L^2([0,\infty))$. Define $F : (0,\infty) \to \mathbb{R}$ by

$$F(x) = \int_0^\infty \frac{f(t)}{1+xt}\, dt\,.$$

(a) Prove that F is continuous.

(b) Prove that F is differentiable.

3. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable and 2π periodic function. Let $\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\, dt$ be the nth Fourier coefficient of f. Prove that the sequence $\big(\hat{f}(n)\big)_{n\in\mathbb{Z}}$ is in $\ell^1(\mathbb{Z})$, i.e.

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty.$$

4. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1(\mathbb{R})$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)g(x)\, dx = g(0)$$

for every $g \in C_0(\mathbb{R})$. Prove that the sequence $(f_n)_{n\in\mathbb{N}}$ is not Cauchy in L^1.

5. Let $g \in L^1([0,1])$ be a function with $\|g\|_1 > 0$.

(a) Let $f \in L^\infty([0,1])$ be a nonnegative function with $\|f\|_\infty > 0$ and such that the set $\{x : f(x) = \|f\|_\infty\}$ has measure zero. Prove that

$$\int fg < \|g\|_1 \, \|f\|_\infty.$$

Give an explicit example of such a function f.

(b) On the other hand, exhibit (i.e. explicitly define) a function $f \in L^\infty([0,1])$ with $\|f\|_\infty > 0$ such that $\int fg = \|g\|_1 \, \|f\|_\infty$.

6. (a) State the Lebesgue Density Theorem.

(b) Let $E \subset \mathbb{R}$ be a measurable set. For $x \in \mathbb{R}$, define $E + x := \{a + x : a \in E\}$. Assume that $E + q = E$ for all $q \in \mathbb{Q}$. Prove that either $m(E) = 0$ or $m(\mathbb{R} \setminus E) = 0$.

Hint. First, try to derive a contradiction assuming that both E and $\mathbb{R}\setminus E$ contain nonempty open intervals. An approximate version of this adapts to the general case.