Math 540 Comprehensive Examination January 21, 2016

Solve five of the following six problems. Each problem is worth 20 points. Calculators, books and notes are not allowed. Good Luck!

<u>Notation:</u>
$\mathbb{Q}$ is the set of rational numbers and $\mathbb{R}$ is the set of real numbers
m is the Lebesgue measure in $\mathbb{R}$
If E is a non-empty Lebesgue measurable set in $\mathbb{R}$ and $p \geq 1$, then $L^p(E)$ denotes Lebesgue's L^p-space

1. Show that for $p > 1$,
$$\lim_{n\to\infty} \int_1^n \frac{\left(1-\frac{t}{n}\right)^n}{t^p} dm(t) = \int_1^\infty \frac{e^{-t}}{t^p} dm(t).$$

2. Let f be a Lebesgue measurable function on the closed interval $[0,1]$. Prove that
(a) $\lim_{p\to\infty} \|f\|_{L^p([0,1])} = \|f\|_{L^\infty([0,1])}$.
(b) Give a counterexample to show that the statement of part (a) fails when $[0,1]$ is replaced by $\mathbb{R}$.

3. A coset of $\mathbb{Q}$ w.r.t. $x \in \mathbb{R}$ in (additive group) $\mathbb{R}$ is the set $x + \mathbb{Q}$. Let E be a set that contains exactly one point from each coset of $\mathbb{Q}$ in $\mathbb{R}$. Prove that
(a) $(r_1 + E) \cap (r_2 + E) = \emptyset$ if $r_1, r_2 \in \mathbb{Q}$ and $r_1 \neq r_2$
(b) $\mathbb{R} = \cup_{r\in\mathbb{Q}}(r + E)$.
(c) Use parts (a) and (b) to prove that if $F \subseteq \mathbb{R}$ is a set such that every subset of F is Lebesgue measurable, then Lebesgue's measure of F is 0 (give a direct proof; reference to Vitali's theorem on existence of Lebesgue non-measurable sets is not sufficient).

4. Let $X = \{1, 2, 3\}$. Let 2^X denote the power set of X, i.e., the set of all subsets of X. Define a set function $\mu^* : 2^X \to [0,2]$ as follows:
$$\mu^*(\varnothing) = 0, \mu^*(X) = 2,$$
$$\mu^*(A) = 1 \text{ if } A \in 2^X, A \neq \varnothing \text{ and } A \neq X.$$
(a) Prove that μ^* is an **outer measure** on X.
(b) Describe all μ^*-measurable sets (in the sense of Carathéodory). Your answer must be specific and different from the definition and its equivalent forms. **Justify your answer.**

5. A function $\varphi : [a,b] \to \mathbb{R}$ is said to be *singular* if
(i) $\varphi \in C[a,b]$ (i.e., φ is continuous on $[a,b]$),
(ii) $\varphi'(x)$ exists a.e. in $[a,b]$,
(iii) $\varphi'(x) = 0$ a.e. in $[a,b]$.
Let $f \in C[a,b] \cap BV[a,b]$ (i.e., f is continuous on $[a,b]$ and is a function of bounded variation on $[a,b]$). Prove that there is an absolutely continuous function $F : [a,b] \to \mathbb{R}$ and a singular function $\varphi : [a,b] \to \mathbb{R}$ such that $f = F + \varphi$.

6. Let $n, q \in \mathbb{N}$ and $q \leq n$. Let $\mathcal{F} = \{E_1, E_2, ...E_n\}$ be a family of Lebesgue measurable subsets of the closed interval $[0,1]$ such that every point $x \in [0,1]$ is an element of at least q sets from the family $\mathcal{F}$. Prove that Lebesgue's measure of at least one set $E_j \in \mathcal{F}$ is not less than q/n.