Math 540 Comprehensive Examination
January 21, 2015

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by m.

1. For each statement, give a counterexample or a short proof/explanation.
a) If $f'(x) = 0$ a.e on $\mathbb{R}$, then f is constant on $\mathbb{R}$.
b) If $f'(x) = 0$ a.e. on $\mathbb{R}$ and f is absolutely continuous on $\mathbb{R}$, then f is constant on $\mathbb{R}$.

2. Suppose that (S_n) is a sequence of measurable subsets of $[0, 1]$, and let

$$S = \{x : x \in S_n \text{ for infinitely many } n\}.$$

(a) Suppose that $\sum_{n=1} m(S_n) < \infty$. Prove that $m(S) = 0$.
(b) Is the conclusion $m(S) = 0$ true if we only assume that $\lim_{n\to\infty} m(S_n) = 0$? Justify your answer.

3. Let m^* be an outer measure defined by

$$m^*(E) = \inf\{m(U) : U \text{ open and } E \subseteq U \subseteq \mathbb{R}^n\}.$$

Suppose that $(E_k)_{k=1}^{\infty}$ is a sequence of subsets of $\mathbb{R}^n$ with $E_k \subseteq E_{k+1}$ for all $k \in \mathbb{N}$. Prove that

$$\lim_{k\to\infty} m^*(E_k) = m^*\left(\bigcup_{k=1}^{\infty} E_k\right).$$

4. Prove or disprove by giving a counterexample the following two propositions (f_n, f are real-valued):

(i) $\left(f_n \overset{n}{\to} f \text{ in } L^2[0,1]\right) \implies \left(f_n^2 \overset{n}{\to} f^2 \text{ in } L^1[0,1]\right)$.

(ii) $\left(f_n \in L^2[0,1],\ f_n \overset{n}{\to} f \text{ weakly in } L^2[0,1]\right) \implies \left(f_n^2 \overset{n}{\to} f^2 \text{ weakly in } L^1[0,1]\right)$.

(Recall that $f_n \overset{n}{\to} f$ weakly in $L^p[0,1]$ iff $\Phi(f_n) \overset{n}{\to} \Phi(f)$ for every Φ linear bounded functional on $L^p[0,1]$.)

5. Assume that $f : \mathbb{R} \to \mathbb{R}$ is nondecreasing,

$$\int_{\mathbb{R}} f'\,dm = 1, \quad \lim_{x\to-\infty} f(x) = 0, \text{ and } \lim_{x\to\infty} f(x) = 1.$$

Prove that f is absolutely continuous on any interval $[a, b]$.

6. Let f_n be a sequence of Lebesgue measurable functions on the interval $[0, 1]$. Assume that f_n converges to a function f m almost everywhere, and that

$$\int_{[0,1]} |f_n|^2 dm \leq 1$$

for each n. Prove that f_n converges to f in L^1.
Hint: Use Egoroff's thm.