Math 540 Comprehensive Examination

January, 2014

Solve five of the following six. Each problem is worth 20 points. Calculators, books and notes are not allowed. The Lebesgue measure is denoted by m.

1. Suppose that a real-valued function f is increasing on $[a, b]$. Prove that

$$\int_a^b f'(t)dt \leq f(b) - f(a).$$

2. Let $f \in L^1(X, \mathcal{A}, \mu)$. Show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any measurable set E with $\mu(E) < \delta$, $\int_E |f| d\mu < \varepsilon$.

3. Let $p \in (1, \infty)$ and $f, g \in L^p(\mathbb{R})$. Suppose that $\|f + g\|_p = \|f\|_p + \|g\|_p$. Find the relation between f and g. Verify your answer!

4. Let $\mathcal{H}_0$ be a closed linear subspace of the Hilbert space $\mathcal{H} = L^2[0, 1]$ and let $f_0 \in \mathcal{H}$. Prove the equality

$$\min_{f \in \mathcal{H}_0} \|f_0 - f\| = \max_{\substack{g \in \mathcal{H}_0^\perp \\ \|g\|=1}} |\langle f_0, g \rangle|.$$

5. Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of $\mathbb{Q}$, and consider the set

$$A = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \left(r_n - \frac{1}{2^{m+n}}, r_n + \frac{1}{2^{m+n}} \right).$$

(i) Is it true that $A = \mathbb{Q}$? Justify your answer.
(ii) Find the measure of this set.

6. Assume that $f \in L^1(\mathbb{R}, m)$ and

$$\left| \int_I f dm \right| \leq [m(I)]^2$$

for any interval I. Prove that $f = 0$ m a.e.