Math 540 Comprehensive Examination
January 16, 2013

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by m.

1. Short answer questions. For each statement give a counterexample or a short proof/explanation.
a) If g is continuous on $\mathbb{R}$ and if $f = g$ m a. e., then f is continuous m a.e. on $\mathbb{R}$.
b) If $f_n \to f$ m a.e. and $\|f_n\|_{L^1(m)} \leq 1$ for each n then $f \in L^1(m)$.
c) If f_n goes to zero pointwise m a.e. on $\mathbb{R}$, then a subsequence goes to zero in measure w.r.t. m.

2. Let $\mathcal{A}$ be a σ-algebra containing all open sets in the interval $I = [0,1]$. Show that $\mathcal{A} \times \mathcal{A}$ contains all open sets in $I \times I$.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space. Suppose that $\{E_n\}$ is a sequence of measurable sets in $\mathcal{A}$ such that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Let E be

$$E = \{x : x \in E_n \text{ for infinitely many } n\}.$$

Prove that $\mu(E) = 0$.

4. Let $K \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} K = 1$. Let $K_\varepsilon(x) = \varepsilon^{-n} K(\varepsilon^{-1} x)$. Prove that if $f \in C_c^m(\mathbb{R}^n)$ for some given positive integer, then $D^\alpha(K_\varepsilon * f)(x)$ converges to $D^\alpha f(x)$ uniformly in $\mathbb{R}^n$. Here $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ for any multi-index $\alpha \in (N \cup \{0\})^n$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq m$.

5. Let $f \in L^1(\mathbb{R}, m)$. Prove that for m a. e. $x \in [0,1]$, the sequence $\{f(n+x)\}_{n=1}^{\infty}$ converges to 0.

6. Prove that

$$Lf = \sum_{n \in \mathbb{Z}} \frac{e^{in}}{2^{|n|}} \widehat{f}(n).$$

gives a bounded linear functional on $L^2([0,1], m)$ and calculate its norm.
Here $\widehat{f}(n) = \langle f, e^{2\pi i n x} \rangle = \int_{[0,1]} f(x) e^{-2\pi i n x} dm(x)$.