Math 540 Comprehensive Examination
January 18, 2012

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by m.

1. Let E and F be Lebesgue measurable subsets of the real line with $m(E) > 0$ and $m(F) > 0$. Show that some translate of E intersects F in a set of positive measure.

2. Prove or disprove the following:
(a) Convergence in measure implies convergence a.e. of a subsequence.
(b) Convergence in L^1 implies convergence in measure.

3. Assume that $f \in L^1(\mathbb{R}, m)$ and

$$\left| \int_I f dm \right| \leq [m(I)]^2$$

for any interval I. Prove that $f = 0$ m a.e.

4. Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}, m)$. Prove that

$$\lim_{\delta \to 0} \int_{\mathbb{R}} |f(x+\delta) - f(x)|^p dm(x) = 0\,.$$

5. Let $p \in (1, \infty)$ and let $\alpha = 1 - \frac{1}{p}$. Assume that f is absolutely continuous on $[0,1]$ and $f' \in L^p([0,1], m)$. Prove that $f \in Lip(\alpha)$, i.e. $\exists C$, $\forall x$, $y \in [0,1]$,

$$|f(x) - f(y)| \leq C|x-y|^\alpha.$$

6. Compute the following limit and justify your calculation:

$$\lim_{n \to \infty} \int_0^\infty \frac{x^{n-2}}{1+x^n} \cos(\pi n x)\, dx.$$