Math 540 Exam

January, 2011

Calculators, books and notes are not allowed!

1. Let m^* be an outer measure defined by

$$m^*(E) = \inf\{m(U) : U \text{ open and } E \subseteq U\} .$$

Here m stands for the Lebesgue measure on $\mathbb{R}$. Prove that for any $E \subseteq \mathbb{R}$, there exists a Lebesgue measurable set G such that $m(G) = m^*(E)$.

2. a) State Fatou's Lemma and the monotone convergence theorem.
b) Use Fatou's lemma to prove the monotone convergence theorem.

3. Let $1 \leq p < \infty$. Suppose that $f_n \to f$ in measure, and $|f_n| \leq g \in L^p$ for all $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} \|f_n - f\|_p = 0$.

4. Let $1 < p < \infty$ and $f, g \in L^p(\mathbb{R}^n)$. Prove that $\|f + g\|_p = \|f\|_p + \|g\|_p$ if and only if there exists a non-negative constant C such that either $f = Cg$ a.e. or $g = Cf$ a.e.

5. Suppose $f : \mathbb{R} \to \mathbb{C}$. F is said to be Lipschitz with constant M if there is a constant M such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$. Prove that f is Lipschitz with constant M iff f is absolutely continuous and $|f'| \leq M$ a.e.

6. Let f_n's and f be measurable complex-valued functions on a measure space $(X, \mathcal{A}, \mu)$. We say that $f_n \to f$ almost uniformly on X if for every $\varepsilon > 0$, there exists $E \subseteq X$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on $X \backslash E$. Prove that if $f_n \to f$ almost uniformly on X, then $f_n \to f$ a.e. and in measure.