# Math 540 Exam

*January, 2010*

*Calculators, books and notes are not allowed!*

**(Do 4 problems out of 5 problems. Mark clearly which 4 problems to be graded. )**

1. Suppose that $f : [0,1] \to \mathbb{R}$ is differentiable a.e..

(a) Show that $f'$ may not be in $L^1([0,1])$ by an example.

(b) If $f' \in L^1([0,1])$, must

$$\int_c^d f'(x)dx = f(d) - f(c)$$

hold for every $[c,d] \subset [0,1]$? Justify your answer.

2. Suppose $\{f_n\}$ is a sequence of bounded measurable functions on a bounded measurable set $K \subset \mathbb{R}$. Prove that

a) If $f_n \to f$ uniformly on $K$, then $\lim_{n\to\infty} \int_K f_n = \int_K f$.

b) Show that the hypothesis "K is bounded" can not be omitted.

3. a) Let $f$ be a measurable function on $[0,1]$. Let $\|f\|_{L^p([0,1])} = \left(\int_{[0,1]} |f(t)|^p dt\right)^{1/p}$ for $1 \le p < \infty$ and

$$\|f\|_{L^\infty([0,1])} = \text{ess sup}_{t\in[0,1]}|f(t)| = \inf\{M : m(\{t \in [0,1] : |f(t)| > M\}) = 0\},$$

where $m$ is Lebesgue measure on $\mathbb{R}$. Prove that

$$\lim_{p\to\infty} \|f\|_{L^p([0,1])} = \|f\|_{L^\infty([0,1])}$$

b) Give a conterexample to show that a) fails when $[0,1]$ is replaced by $\mathbb{R}$.

4. Suppose that $(A_n)$ is a sequence of measurable subsets of $[0,1]$, and let

$$A = \{x : x \in A_n \text{ for infinitely many } n\}.$$

(a) Suppose that $\sum_{n=1} m(A_n) < \infty$. Prove that $m(A) = 0$.

(b) Is the conclusion $m(A) = 0$ true if we only assume that $\lim_{n\to\infty} m(A_n) = 0$? Prove your answer.

5. A linear operator $T$ on $L^2(\mathbb{R})$ is an operator on $L^2(\mathbb{R})$ such that $T(\alpha f + \beta g) = \alpha f + \beta g$ for all $f, g \in L^2(\mathbb{R})$ and all $\alpha, \beta \in \mathbb{C}$. An operator $T$ is called a weak $(2,2)$ operator if

$$m(\{x \in \mathbb{R} : |Tf(x)| > \lambda\}) \le \frac{C\|f\|^2_{L^2(\mathbb{R})}}{\lambda^2},$$

holds for any $f \in L^2$ and any $\lambda$, where the constant $C$ is independent of $f$ and $\lambda$, and $m$ is the Lebesgue measure on $\mathbb{R}$. For any $\epsilon > 0$, let $T_\epsilon$ be a linear operator on $L^2(\mathbb{R})$ and define

$$T^*f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|.$$

Let $A$ be the set defined by

$$A = \{f \in L^2(\mathbb{R}) : \lim_{\epsilon\to 0} T_\epsilon f(x) = f(x) \ a.e\}.$$

Prove that if $T^*$ is weak $(2,2)$, then $A$ is closed in $L^2(\mathbb{R})$.