Math 540 Comprehensive Examination

January 12, 2009

Solve problems (1),(2), and (3). Also solve **two** of the problems (4),(5),(6), and (7).

Part I (Solve all problems)

(1) (20=10+10 pts)

(a) Assuming Fatou's lemma prove Lebesgue's Dominated Convergence Theorem.

(b) Prove that Lebesgue's Dominated Convergence Theorem holds when pointwise convergence is replaced by convergence in measure.

(2) (20 pts) For a nonnegative measurable function f on a measure space (X, μ), let

$$\omega_f(t) := \mu(\{x \in X : f(x) > t\}).$$

Prove that for every $p \in [1, \infty)$,

$$f \in L^p(\mu) \iff \sum_{k \in \mathbb{Z}} 2^{kp} \omega_f(2^k) < \infty.$$

(3) (20=4 × 5 pts) State whether each of the following assertions is true or false. Give a short proof or counterexample.

(a) Every dense open subset of $(0, 1)$ has Lebesgue measure one.

(b) If $f : \mathbb{R} \to \mathbb{R}$ is equal to a continuous function $g : \mathbb{R} \to \mathbb{R}$ almost everywhere, then f is continuous at almost every point $x \in X$.

(c) If $f \in AC[0, 1]$, then f is Lipschitz on $[0, 1]$.

(d) If f is Riemann integrable on $[a, b]$, then $|f|^\alpha$ is Riemann integrable on $[a, b]$ for every $\alpha > 0$.

Part II (Solve <u>two</u> of the following four)

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere, with $\sup_{x \in \mathbb{R}} |f'(x)| \leq K$. Show that

$$m(f(E)) \leq K m(E)$$

for all measurable sets $E \subseteq \mathbb{R}$. (m denotes the Lebesgue measure on $\mathbb{R}$.)

(5) (20=5+15 pts) (a) Give an example of $f \in BV[0, 1]$ for which

$$\lim_{x \to 1^-} T_0^x(f) \neq T_0^1(f).$$

(b) Under the additional hypothesis that f is continuous at 1, show that

$$\lim_{x \to 1^-} T_0^x(f) = T_0^1(f).$$

(6) (20 pts) Let $f : \mathbb{R} \to \mathbb{R}$. Prove that f satisfies for some $M > 0$ the Lipschitz condition

$$|f(x) - f(y)| \leq M|x - y|, \quad \forall x, y \in \mathbb{R},$$

if and only if f is absolutely continuous and $|f'(x)| \leq M$ a.e.

(7) (20 pts) Let $1 < p < \infty$ and $0 < \alpha \leq 1 - 1/p$. Compute

$$\sup \left\{ \left| \int_{[0,1]} x^{-\alpha} g(x)\, dx \right| : g \text{ measurable and } \int_{[0,1]} |g(x)|^p dx \leq 1 \right\},$$

and find a function $g \in L^p[0,1]$ for which the supremum is attained, when finite.