Math 540 Comprehensive Examination

January 19, 2007

Solve both problems in Part I and choose 3 out of 5 problems in Part II. Indicate your choice. In Part II credit will be given only for three problems.

m stands for the Lebesgue measure on $\mathbb{R}$.

Good luck

Part I

1) Answer the following questions. Give short justifications for your answers.
   (a) (5 pts) State Fatou's Lemma and give an example of strict inequality.
   (b) (5 pts) Give an example of a Lebesgue integrable not Riemann integrable function on $[0, 1]$.
   (c) (5 pts) Give an example of a continuous not absolutely continuous function on $[0, 1]$.
   (d) (5 pts) Recall that a set $A \subset \mathbb{R}$ is measurable if

$$m^*(B) = m^*(A \cap B) + m^*(A^c \cap B)$$

   holds for every B and the outer measure m^*. Here $A^c = \mathbb{R} \setminus A$ is the complement. Show that $[0, 1]$ is measurable.

2) Let $f \in L^1(\mathbb{R})$ and $a > 0$.
   (a) (10 pts) Show that $\lim_{n\to\infty} n^{-a} f(nx) = 0$ holds for almost every $x \in \mathbb{R}$. (Hint: look at $\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} |n^{-a} f(nx)| dx$).
   (b) (10 pts) Show that the series $\sum_{n=-\infty}^{\infty} f(n + \frac{x}{a})$ converges absolutely for almost every $x \in \mathbb{R}$.

Part II

3) Let $A \subset \mathbb{R}$ be a measurable set with $m(A) < \infty$.
   (a) (10 pts) Prove that the function $f : \mathbb{R} \to [0, \infty)$, $f(x) = m(A \cap (-\infty, x])$ is continuous.
   (b) (10 pts) Let $0 < b < m(A)$. Show that there exists a measurable subset $B \subset A$ such that $m(B) = b$. (In the literature one says that m is atomless.)

4) (20 pts) Show that

$$\lim_n \int_0^1 \frac{1 + nx^2 + n^2x^4}{(1 + x^2)^n} \, dx$$

exists and determine the limit.

5) (20 pts) Let $1 \leq p < \infty$. Let (f_n) be a sequence in $L^p(\mathbb{R})$ which converges almost everywhere to $f \in L^p(\mathbb{R})$ and

$$\lim_n \int_{\mathbb{R}} |f_n|^p dm = \int_{\mathbb{R}} |f|^p dm.$$

Show that f_n converges to f with respect to the L^p norm.

(Hint: Modify the proof of the dominated convergence theorem using $h_n = |f_n - f|^p \leq g_n = 2^{p-1}(|f_n|^p + |f|^p)$; why is this inequality true?).

6) (a) (10 pts) Let (f_n) be a sequence in $L^2[0,1]$ and $f \in L^2[0,1]$ such that f_n converges to f almost everywhere. Show that

$$\lim_n \|f_n - f\|_2^2 = 0 \iff \lim_n \|f_n\|_2 = \|f\|_2.$$

(b) (10 pts) Show that $L^4[0,1]$ is not a Hilbert spaces.

7) (a) (5 pts) Show that the product of two absolutely continuous functions is absolutely continuous.

(b) (15 pts) Let f be a function of bounded variation on $[a,b]$. Show that

$$\int_a^b |f'| dm \leq T_a^b(f),$$

where $T_a^b(f) = \sup_{\sigma=\{a=x_0,\dots,x_n=b\}} \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)|$ is the total variation. The supremum is taken over all partitions σ of $[a,b]$.