Math 540 Comprehensive Examination

January 24, 2007

Solve all four problems in Part I and choose one problem in Part II. Indicate your choice. Credit will be given only for <u>one</u> problem in Part II. Each problem is worth 20 points.

m denotes the Lebesgue measure on $\mathbb{R}$.

Part I

1. Denote by $[x]$ the integer part of x. Consider the function

$$f : [0,1] \to \mathbb{R}, \qquad f(x) = \sum_{n=1}^{\infty} \frac{[nx]}{3^n}.$$

(a) Prove that f is Lebesgue integrable and compute its integral on $[0,1]$, expressing the answer in the form of a rational number.

(b) Is f Riemann integrable on $[0,1]$? Justify your answer.

2. Decide whether each of the following three statements is true or false. Justify your answer (i.e. provide the reason and a short proof if true and a counterexample if false).

(a) Let (f_n) be a sequence of measurable functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n \to f$ pointwise on $\mathbb{R}$. Then there exists a subsequence (f_{n_k}) such that $f_{n_k} \to f$ in measure.

(Recall that, by definition, $f_n \to f$ in measure when $m\{|f_n - f| > \delta\} \xrightarrow{n} 0$ for every $\delta > 0$.)

(b) If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and f' is bounded (a.e.) on (a,b), then f must be absolutely continuous.

(c) Let $1 \le p < \infty$ and $f \in L^p(\mathbb{R})$. Then

$$\forall \varepsilon > 0,\ \exists N > 0 \text{ such that } m(\{|f| > N\}) < \varepsilon.$$

3. Let

$$f(x) = \int_1^{\infty} \frac{e^{-xy}}{y^3}\, dy, \qquad x > 0.$$

Show that f is differentiable on $(0,\infty)$ and find a formula for f'.

4. Suppose that B is a measurable subset of $[0,1]$. Define

$$g(x) = m\big((-\infty, x) \cap B\big).$$

Prove that g' exists almost everywhere and calculate $\int_{[-1,2]} g'\, dm$ explaining your reasoning.

Part II

5. Let $f : [0,1] \to [0,\infty)$ in L^1 such that

$$(*) \qquad \int_E f\, dm \leq \sqrt{m(E)} \quad \text{for every } E \subseteq [0,1] \text{ measurable.}$$

(a) Prove that $f \in L^p[0,1]$ for all $p \in [1,2)$.
(b) Give an example of a function f which satisfies $(*)$ but $f \notin L^2[0,1]$.

6. In a Hilbert space $\mathscr{H}$ consider an orthonormal set $(u_n)_{n=1}^{\infty}$.
(a) Prove Bessel's inequality

$$(**) \qquad \sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 \leq \|x\|^2, \qquad \forall x \in \mathscr{H}.$$

(b) Use part (a) to show that if, in addition, $\{u_n : n \geq 1\}^{\perp} = \{0\}$, then equality holds in $(**)$.