Math 540 Comprehensive Examination
February 7, 2006

Answer any 5 out of 6. Indicate which 5 problems you want graded. All problems have equal value.

m denotes the Lebesgue measure on $\mathbb{R}$.

I. Suppose $f : [0,1] \to \mathbb{R}$ is an integrable function. Prove that

$$\lim_{h\to 0} \int_{[0,1]} \frac{|1+hf(t)|-1}{h}\, dm(t) = \int_{[0,1]} f(t)\, dm(t).$$

II. Suppose $f \in L^1([0,1])$. Prove that for every $\varepsilon > 0$ there is $\delta > 0$ such for every measurable set E with $mE < \delta$ we have $\int_E |f|\, dm < \varepsilon$. Conclude that the function F defined by $F(x) = \int_{[0,x]} f\, dm$ is absolutely continuous on $[0,1]$.

III. Decide whether each of the following statements is true or false and justify your answer.

(a) If $f : [0,\infty) \to \mathbb{R}$ is continuous and $\lim\limits_{x\to\infty} f(x)$ exists and is finite, then f is uniformly continuous on $[0,\infty)$.

(b) Let $f_n : [0,1] \to [0,\infty)$ be a sequence of integrable functions. If $f_n \to f$ in measure on $[0,1]$, then

$$\int_{[0,1]} f\, dm \leq \liminf_{n\to\infty} \int_{[0,1]} f_n\, dm.$$

(c) If $f : [0,1] \to \mathbb{R}$ is a function such that $|f(x) - f(y)| \leq |\sqrt{x} - \sqrt{y}|$ for all $x, y \in [0,1]$, then f is absolutely continuous on $[0,1]$.

IV. Let $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x) \sin(nx)\, dm(x) = 0.$$

V. (a) Suppose $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Prove that $f \in L^p(\mathbb{R})$ for every $1 \leq p < \infty$.

(b) Is it true that $L^1(\mathbb{R}) \cap L^3(\mathbb{R}) \subseteq L^2(\mathbb{R})$? Justify your answer.

VI. In the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N})$ a sequence (f_n) is said to *converge weakly* to f if $\lim\limits_n \langle f_n, g\rangle = \langle f, g\rangle$ for all $g \in \mathcal{H}$.

(a) Prove that every infinite orthonormal set $(u_n)_n$ in $\mathcal{H}$ converges weakly to zero.

(b) Prove that if $f \in \mathcal{H}$, $\|f\| < 1$, then there is a sequence $(f_n)_n$ in $\mathcal{H}$, $\|f_n\| = 1$, such that $f_n \to f$ weakly in $\mathcal{H}$.