Math 540 Comprehensive Examination
August 21, 2017

Do five out of six problems. Each problem is worth 20 points. Justify all claims.

Notation. For a set $A \subseteq X \times Y$ and points $x \in X, y \in Y$, let

$$A_x := \{v \in Y : (x, v) \in A\} \text{ and } A^y := \{u \in X : (u, y) \in A\}.$$

Below λ denotes the Lebesgue measure on $\mathbb{R}$.

1. (a) Let $p > 1$ and let $f_n : (0,1) \to (0,\infty)$ be a sequence of measurable functions such that $x(f_n(x))^p \leq 1$ for all $0 < x < 1$ and all $n \in \mathbb{N}$, and $f_n \to 0$ λ-a.e. Prove that

$$\int_0^1 f_n \, d\lambda \to 0.$$

(b) Is the result of part (a) true when $p = 1$? Prove or give a counterexample.

2. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and $f : X \to [0, \infty]$ a non-negative measurable function. Prove:

$$\int f \, d\mu = \int_0^\infty \mu(\{x \in X : y \leq f(x)\}) \, d\lambda(y).$$

3. Decide whether each of the following statements is true or false. Justify your answer with a short proof if the statement is true or a counterexample if it is false.

(a) Let (f_n) be a sequence in $L^p([0,1], \lambda)$ and let $f \in L^p([0,1], \lambda)$. If $f_n \to_{L^p} f$ (i.e. converges in the L^p-norm), then $f_n \to_\lambda f$ (i.e. converges in measure).

(b) If $f : [0,1] \to \mathbb{R}$ is absolutely continuous and one-to-one, then f^{-1} is absolutely continuous.

(c) If $f : [0,1] \to \mathbb{R}$ is a Lebesgue integrable function with $\int_A f \, d\lambda = 0$ for all Lebesgue measurable sets $A \subseteq [0,1]$, then $f = 0$ λ-a.e.

(d) If $f : [0,1] \to \mathbb{R}$ is continuous, then f is of bounded variation.

4. Suppose that $f : [0,1] \to [0,\infty)$ is in $L^1([0,1], \lambda)$ and that, for every Lebesgue measurable $A \subseteq [0,1]$,

$$\int_A f \, d\lambda \leq \sqrt{\lambda(A)}.$$

Prove that $f \in L^p([0,1], \lambda)$ for all $p \in [1, 2)$.

Hint. For each integer $n \geq 0$, consider $A_n := \{x \in [0,1] : 2^n \leq f < 2^{n+1}\}$.

5. Let $c > 0$.

(a) Let $A \subseteq \mathbb{R}^2$ be the set defined by $A_x := (x, x+c]$ for each $x \in \mathbb{R}$. Show that A is Borel and explicitly describe A^y for each $y \in \mathbb{R}$.

(b) Let $f : \mathbb{R} \to [0,$ be a right-continuous non-decreasing bounded function, so $f(\infty) := \lim_{x \to \infty} f(x)$ and $f(-\infty) := \lim_{x \to -\infty} f(x)$ exist. Assuming $f(-\infty) = 0$, prove that

$$\int_{\mathbb{R}} [f(x+c) - f(x)] \, d\lambda(x) = c f(\infty).$$

6. Let f be an absolutely continuous (2π)-periodic function on $\mathbb{R}$ such that

$$0 = \int_{[0,2\pi]} f(x) \, dx = \int_{[0,2\pi]} f(x) e^{ix} \, dx = \int_{[0,2\pi]} f(x) e^{-ix} \, dx.$$

Prove that $\|f\|_2 \leq \frac{1}{2} \|f'\|_2$.