# Math 540 Real Analysis — Comprehensive Exam — August 2018

Do five of the six problems. Each problem is worth 20 points. Justify all claims.

*Notation:*
$n$ is a positive integer.
$m$ is Lebesgue measure on $\mathbb{R}$, and $m_n$ is Lebesgue measure on $\mathbb{R}^n$.

***Problem* 1.** Show that if $f \in L^1(\mathbb{R}^n)$, and $g$ is continuous and bounded on $\mathbb{R}^n$, then the convolution $f * g(x)$ is continuous.

***Problem* 2.** Suppose $f_k \to f$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$, and define $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{if } |x| \leq 1, \\ 1/|x| & \text{if } |x| > 1. \end{cases}$$

Show that if $1 \leq p < n/(n-1)$ then $fg \in L^1(\mathbb{R}^n)$ and $\int f_k g\, dm_n \to \int fg\, dm_n$ as $k \to \infty$.

***Problem* 3.** Let $(X, \mathcal{M}, \mu)$ be a measure space and $(f_n)$ be a sequence of measurable functions such that $f_n \geq 0$ on $X$ for every $n$ and $f_n$ converges in measure to $f$. Show that

$$\int_X f\, d\mu \leq \liminf_n \int_X f_n\, d\mu.$$

***Problem* 4.** Show that the function

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

is not Lebesgue integrable on the unit square $[0,1]^2$.

***Problem* 5.** Let $f_n : [a,b] \to \mathbb{R}$ be an increasing function, for each $n$. Suppose the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges to a real number for each $x \in [a,b]$, that is, $f : [a,b] \to \mathbb{R}$. Prove that

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) \quad \text{for almost every } x \in [a.b].$$

***Problem* 6.** Suppose $f(t)$ is $2\pi$-periodic and absolutely continuous, so that $f'$ exists and integration by parts is valid. Assume $f, f' \in L^2[-\pi,\pi]$ with $\int_{-\pi}^{\pi} f(t)\, dt = 0$.

Prove

$$\int_{-\pi}^{\pi} |f(t)|^2\, dt \leq \int_{-\pi}^{\pi} |f'(t)|^2\, dt,$$

and find all functions $f$ for which equality holds.