Math 540 Comprehensive Examination August 16, 2016

Solve five of the following six problems. Each problem is worth 20 points. Calculators, books and notes are not allowed. Good Luck!

Notation: m is the Lebesgue measure on $\mathbb{R}$.

1. Let $(a_j)_{j=1}^{\infty}$ be a sequence of complex numbers. Prove that
$$\lim_{q\to\infty}\left(\sum_{j=1}^{n}|a_j|^q\right)^{1/q}=\sup_{j\in\{1,\cdots,n\}}|a_j|$$
provided that $n\in\mathbb{N}$ is independent of q.

2. Show that
$$\lim_{b\to\infty}\int_0^b\frac{\sin x}{x}dx=\frac{\pi}{2},$$
by integrating $e^{-xy}\sin x$ with respect to x and y. **Justify your answer.**

3. Let f be a measurable function on a measure space $(X,\mathcal{A},\mu)$. The *decreasing rearrangement* of f is the function $f^*:(0,\infty)\to[0,\infty]$ defined by
$$f^*(t)=\inf\{\alpha:\mu(\{x:|f(x)|>\alpha\})\le t\},$$
where $\inf\varnothing$ is defined to be ∞. Prove that
(a) f^* is decreasing;
(b) if $f^*(t)<\infty$, then $\mu(\{x:|f(x)|>f^*(t)\})\le t$.

4. Let $f:[0,1]\to\mathbb{R}$ be a continuous function and $g:[0,1]\to[0,1]$ be a Borel function. Given $n\in\mathbb{N}$, explain why the function $f\left[g(x)^n\right]$ is Lebesgue integrable over the closed interval $[0,1]$. Compute the limit
$$\lim_{n\to\infty}\int_0^1 f\left([g(x)]^n\right)dx.$$
Justify your answer.

5. If $E\subseteq[0,1]$, then $\chi_E:[0,1]\to\mathbb{R}$ denotes the characteristic function of E. Find necessary and sufficient condition for the set $E\subseteq[0,1]$ to be Riemann integrable over the closed interval $[0,1]$. **Justify your answer.**

6. Let $E\subseteq\mathbb{R}$ be a non-empty Lebesgue measurable set and $(f_n:E\to\mathbb{R})_{n=1}^{\infty}$ be a sequence of Lebesgue measurable functions. We say that the sequence $(f_n)_{n=1}^{\infty}$ *converges almost uniformly* to a measurable function $f:E\to\mathbb{R}$ on the set E as $n\to\infty$ if for each $\varepsilon>0$, there is a non-empty Lebesgue measurable set $E_\varepsilon\subseteq E$ such that $m(E_\varepsilon)<\varepsilon$ and the sequence $(f_n)_{n=1}^{\infty}$ converges to f uniformly on the set $E'_\varepsilon=E\backslash E_\varepsilon$ as $n\to\infty$. Suppose that $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in Lebesgue's measure m, that is, for every $\varepsilon>0$,
$$\lim_{n,j\to\infty}m\{x\in E:|f_n(x)-f_j(x)|\ge\varepsilon\}=0.$$
Prove that there is a Lebesgue measurable function $f:E\to\mathbb{R}$ and that the sequence $(f_n)_{n=1}^{\infty}$ has a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that $(f_{n_k})_{k=1}^{\infty}$ converges almost uniformly to f on the set E as $k\to\infty$.