Math 540 Comprehensive Examination

August 19, 2015

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by m. Calculators, books and notes are not allowed.

1. Let $1 \le p < \infty$. Suppose $f_n \to f$ in measure and $|f_n| \le g \in L^p$ for all n. Prove that $\lim_{n\to\infty} \|f_n - f\|_p = 0$.

2. Let m be Lebesgue measure on $\mathbb{R}$. Suppose that $\int_{\mathbb{R}} |f| dm = c \in (0,\infty)$ and $\alpha \in (0,1)$. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} n \log\left(1 + (|f|/n)^{\alpha}\right) dm = \infty .$$

3. Suppose that f is a continuous function on $\mathbb{R}$, with period 1. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\theta) = \int_0^1 f(t)dt$$

for every irrational number $\theta \in \mathbb{R}$.

4. Fix $1 \le p < \infty$.

i) Assume that f is absolutely continuous on every compact interval, and $f' \in L^p(\mathbb{R}, m)$. Prove that

$$\sum_{n\in\mathbb{Z}} |f(n+1) - f(n)|^p < \infty.$$

ii) Prove or give a counterexample: The statement above remains valid if we instead assume that f is continuous, of bounded variation on every compact interval, and $f' \in L^p(\mathbb{R}, m)$.

5. Let $f(x) = x\sin(1/x)$, $x \in \mathbb{R}\backslash\{0\}$, $f(0) = 0$.

i) Is f uniformly continuous on $\mathbb{R}$? Justify your claim in detail.

ii) Is f of bounded variation on $\mathbb{R}$? Justify your claim in detail.

6. i) Fix a constant $0 < c < 1$. Let $f \in L^1([0,1], m)$. Assume that for every measurable $A \subset [0,1]$ with $m(A) = c$, we have $\int_A f dm = 0$. Prove that $f = 0$ almost everywhere.

ii) Prove by giving a counterexample that the statement above is false if we replace "measurable" with "interval".