Math 540 Comprehensive Examination

August 21, 2014

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by m.

1. Prove or disprove the following:
(a) Convergence in measure implies convergence a.e. of a subsequence.
(b) Convergence in L^1 implies convergence in measure.

2. Let f be a function on $[a, b]$ of total variation $T_a^b f < \infty$.

(i) Prove that $\displaystyle\int_{[a,b]} |f'| \leqslant T_a^b f$.

(ii) Prove that if f is absolutely continuous then equality holds in (i).

3. Prove that

$$Lf = \sum_{n\in\mathbb{Z}} \frac{e^{in}}{2^{|n|}} \widehat{f}(n).$$

gives a bounded linear functional on $L^2([0,1], m)$ and calculate its norm.
Here $\widehat{f}(n) = \langle f, e^{2\pi i n x}\rangle = \int_{[0,1]} f(x) e^{-2\pi i n x} dm(x)$.

4. Let g be a bounded measurable function and f be integrable over $\mathbb{R}$. Show that

$$\lim_{t\to 0} \int_{\mathbb{R}} |g(x)| \big| f(x) - f(x+t) \big| dx = 0\,.$$

5. Let ω be an irrational number. For any $f \in L^2(\mathbb{T})$, show that

$$\lim_{N\to 0} \int_{\mathbb{T}} \Big| \frac{1}{N} \sum_{n=1}^{N} f(x + n\omega) - \int_{\mathbb{T}} f(\theta) d\theta \Big|^2 dx = 0\,.$$

6. Prove that if $E \subset \mathbb{R}$ is a set such that every subset of E is Lebesgue measurable, then $m(E) = 0$.