Math 540 Comprehensive Examination
August 20, 2013

Solve five of the following six. Each problem is worth 20 points. The Lebesgue measure is denoted by m.

1. Let μ^* be an outer measure on X and $\mu^*(X) < \infty$. We call $E \subseteq X$ regular if there exists a μ^*-measurable set A with $E \subseteq A$ and $\mu^*(A) = \mu^*(E)$. Suppose E is a regular set. Prove that E is μ^*-measurable iff $\mu^*(E) + \mu^*(E^c) = \mu^*(X)$.

2. Let E be a Lebesgue measurable set with $m(E) > 0$. Show that $E - E$ contains an open interval centered at 0.

3. Let f_n be a sequence of Lebesgue measurable functions on the interval $[0,1]$. Assume that f_n converges to a function f m almost everywhere, and that

$$\int_{[0,1]} |f_n|^2 dm \leq 1$$

for each n. Prove that f_n converges to f in L^1.
Hint: Use Egoroff's thm.

4. Prove or disprove the following:
a) Convergence in measure implies convergence a.e. of a subsequence.
b) Convergence in L^1 implies convergence in measure.

5. a) Provide a non-constant function f such that f is differentiable everywhere in $[-1,1]$ and $f \in BV([-1,1])$.
b) Provide a function f such that f is differentiable everywhere in $[-1,1]$ but $f \notin BV([-1,1])$.

6. Let $1 \leq r < p < s < \infty$.
a) Show that

$$L^r(\mathbb{R}, m) \cap L^s(\mathbb{R}, m) \subset L^p(\mathbb{R}, m).$$

b) Show that

$$L^s([0,1], m) \subset L^r([0,1], m).$$

c) Is it true that $L^s(\mathbb{R}, m) \subset L^r(\mathbb{R}, m)$? Justify your answer.