Math 540 Exam

August, 2011

Calculators, books and notes are not allowed!

1. Let μ be counting measure on $\mathbf{N}$. Interpret Fatou's lemma and the dominated convergence theorems for μ as statements about infinite series.

2. Let $f \in L^1(m)$, where m is Lebesgue measure on $\mathbb{R}$. Suppose

$$F(x) = \int_{-\infty}^{x} f\, dm,$$

then show that F is uniformly continuous on $\mathbb{R}$.

3. Suppose that $\{f_n\}$ be a sequence of measurable functions on measure space $(X, \mathcal{A}, \mu)$, and $f_n \to f$ in measure. If there exists $g \in L^1(X, \mathcal{A}, \mu)$ such that $|f_n| \le g$ holds for all n, then prove that

$$\lim_{n\to\infty} \int f_n = \int f.$$

4. a) State Fubini-Tonelli theorem for σ-finite measure spaces.
b) Give an example to show the hypothesis of σ-finiteness in Fubini-Tonelli is necessary.

5. Suppose that a real-valued function f is increasing on $[a, b]$. Prove that

$$\int_a^b f'(t)dt \le f(b) - f(a).$$

6. Let f be a measurable function on a measure space $(X, \mathcal{A}, \mu)$. The decreasing rearrangement of f is the function $f^* : (0, \infty) \to [0, \infty]$ defined by

$$f^*(t) = \inf\{\alpha : \mu(\{x : |f(x)| > \alpha\}) \le t\},$$

where $\inf \emptyset$ is defined to be ∞. Prove that
a) f^* is decreasing;
b) if $f^*(t) < \infty$, then $\mu(\{x : |f(x)| > f^*(t)\}) \le t$.