# Math 540 Exam

*August, 2010*

*Calculators, books and notes are not allowed!*

1. Let $E, F$ be measurable subsets of the real line with $m(E) > 0$ and $m(F) > 0$. Here $m$ is the Lebesgue measure on $\mathbb{R}$. Show that $F \cap$ (some translate of $E$) is a set of positive measure.

2. Let $F : [0, \infty) \mapsto [0, \infty)$ be a continuously differentiable and monotone function with $F(0) = 0$. Prove that

$$\int_{\mathbb{R}} F(f(x))dm = \int_0^\infty F'(t)m\{x \in \mathbb{R} : f(x) > t\}dt\,,$$

where $f \geq 0$ and $f \in L^1(\mathbb{R})$.

3. a) Let $g$ be an integrable function on $[0, 1]$. Show that there is a bounded measurable function $f$ such that $\|f\|_\infty \neq 0$ and

$$\int_{[0,1]} f(x)g(x)dm = \|f\|_\infty \|g\|_1\,.$$

b) Let $g$ be a bounded measurable function. Show that for each $\varepsilon > 0$, there is an integrable function $f$ such that $\|f\|_1 \neq 0$ and

$$\int f(x)g(x)dm \geq (\|g\|_\infty - \varepsilon)\|f\|_1\,.$$

4. Let $1 \leq p \leq \infty$. Suppose $f \in L^p(\mathbb{R})$ and $g \in L^{p'}(\mathbb{R})$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Prove that $f * g(x) = \int f(x - y)g(y)dy$ is a bounded continuous function on $\mathbb{R}$.

5. a) Give an example of a function that has a positive derivative at a point, but is not monotone increasing in any neighborhood of that point.
b) Give an example of a function of two variables which has all directional derivatives at a point of discontinuity.

6. Let $\omega$ be an irrational number and let $\mathbb{T}$ be the circle of length 1 centered at the origin. For any $f \in L^2(\mathbb{T})$, show that

$$\lim_{N\to 0} \int_{\mathbb{T}} \left| \frac{1}{N} \sum_{n=1}^{N} f(x + n\omega) - \int_{\mathbb{T}} f(\theta)d\theta \right|^2 dx = 0\,.$$

Note that $f$ is defined to have period 1.