Math 540 Comprehensive Examination
August 17, 2009

Solve (1), (2) and (3). Also also solve two of problems (4), (5) and (6). In the following, m stands for the Lebesgue measure on $\mathbb{R}$.

Part I (Solve all three problems, total 60 points)

(1) (20 = 10 + 10 points)

Let f be a nondecreasing real valued function on $[a, b]$.

(i) Show that f has at most countably many discontinuous points.

(ii) Show that f is measurable on $[a, b]$.

(2) (20 = 10 + 10 points)

State whether each of the following assertions is true or false. Give a proof or counter-example.

(i) Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of measurable functions such that $f_n \to f$ pointwise on $\mathbb{R}$. Then there exists a subsequence $f_{n_k} \to f$ in measure.

(ii) Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{R})$. Then for arbitrary $\varepsilon > 0$, there exists a sufficiently large $N > 0$ such that

$$m(\{x \in \mathbb{R} : |f(x)| > N\}) < \varepsilon.$$

(3) (20 points)

Let $f(x) = \int_1^\infty \frac{e^{-xy}}{y^3} dy$. Show that $f(x)$ is is differentiable on $(0, \infty)$ and find a formula for $f'(x)$.

Part II (Solve two from the following three, total 40 points)

(4) (20 points)

Let $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous on finite closed intervals and assume that for $1 < p < \infty$, $f' \in L^p(\mathbb{R})$. Show that

$$\sum_{n=-\infty}^{+\infty} |f(n+1) - f(n)|^p < \infty.$$

(5) (20 points)

Suppose that f is a positive measurable function on $[0,1]$ satisfying

$$m\{x \in [0,1] : f(x) > y\} \leq \frac{1}{y(\log y)^2}$$

for all $y > 2$. Prove that f is Lebesgue integrable.

(6) (20 points)

Let $\alpha < \beta$ and (f_n) be a sequence of measurable functions on $[0,1]$ such that

$$\int_{[0,1]} \sup_n n^\beta |f_n(x)| dm(x) < \infty.$$

Show that

$$F_k(x) = \sum_{n=1}^{2^k} n^\alpha f_n(x)$$

converges almost everywhere on $[0,1]$.