Math 540 Comprehensive Examination

August 18, 2008

Solve problems (1),(2),(3) and (4). Also solve one of the problems (5) and (6). In the following m stands for the Lebesgue measure on $\mathbb{R}$.

Good luck

Part I (Solve all problems)

(1) (20=10+10 pts)

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. Assume that

$$\exists M > 0,\ \exists c \in (0,1),\ \forall \varepsilon > 0, \quad m\{|f| \geqslant \varepsilon\} \leqslant \frac{M}{\varepsilon^c}.$$

Prove that f is Lebesgue integrable.

(ii) Prove or disprove by counterexample: $\bigcap_{1\leqslant p<\infty} L^p(\mathbb{R}) \subset L^\infty(\mathbb{R})$.

(2) (20 pts) Prove the Riemann-Lebesgue lemma:

$$\forall f \in L^1(\mathbb{R}), \quad \lim_{|n|\to\infty} \int_{-\infty}^{\infty} f(x)\cos(nx)\,dx = 0.$$

(3) (20=10+10 pts) Suppose that the function $f : [a,b] \to [c,d]$ is absolutely continuous and bijective.

(i) Must it be true that $f^{-1} : [c,d] \to [a,b]$ is absolutely continuous ?

(ii) If $E \subset [c,d]$ and $m(E) = 0$, must it be true that $m\big(f^{-1}(E)\big) = 0$?

(4) (20=10+10 pts) Denote $L^p = L^p(\mathbb{R})$.

(i) Let $1 \leqslant p \leqslant r \leqslant q < \infty$. Prove that

$$L^p \cap L^q \subset L^r.$$

(ii) Let $f \in \bigcap_{1\leqslant p<\infty} L^p$. Prove that the function $p \mapsto p \ln \|f\|_p$ is convex on $[0,\infty)$.

Part II (Solve <u>one</u> of the following three)

(5) (20=10+10 pts)

(i) Compute the following integral, justifying all steps:

$$I_a = \int_{-\infty}^{\infty} e^{-ax^2}\,dx, \qquad a > 0.$$

(ii) Compute the sum of the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

justifying all steps.

(6) (20=8+12 pts) Let $\mathcal{M}$ denote the space of real-valued measurable functions on $[0,1]$ (identifying any two functions which are equal almost everywhere). Denote

$$\rho(g) = \int_{[0,1]} \frac{|g|}{1+|g|}\, dm, \qquad g \in \mathcal{M}.$$

(i) Show that

$$d(f,g) = \rho(f-g)$$

defines a metric on $\mathcal{M}$.

(ii) Let $f, f_n \in \mathcal{M}$. Show that

$$f_n \to f \text{ in measure} \iff \lim_n d(f_n, f) = 0.$$