Math 540 Comprehensive Examination

August 18, 2007

Solve five of the following six. Each problem is worth 20 points.

1. Compute the following limit and justify your calculation:

$$\lim_{n\to\infty} \int_0^\infty \frac{x^{n-2}}{1+x^n} \cos(\pi n x)\, dx.$$

2. (10+10 pts.) Prove or disprove by giving a counterexample the following two propositions (f_n, f are real-valued):

(i) $\left(f_n \xrightarrow{n} f \text{ in } L^2[0,1]\right) \Longrightarrow \left(f_n^2 \xrightarrow{n} f^2 \text{ in } L^1[0,1]\right)$.

(ii) $\left(f_n \in L^2[0,1],\ f_n \xrightarrow{n} f \text{ weakly in } L^2[0,1]\right) \Longrightarrow \left(f_n^2 \xrightarrow{n} f^2 \text{ weakly in } L^1[0,1]\right)$.

(Recall that $f_n \xrightarrow{n} f$ weakly in $L^p[0,1]$ iff $\Phi(f_n) \xrightarrow{n} \Phi(f)$ for every Φ linear bounded functional on $L^p[0,1]$.)

3. (10+10 pts.) Let $\{r_n\}_{n=1}^\infty$ be an enumeration of $\mathbb{Q}$, and consider the set

$$A = \bigcap_{m=1}^\infty \bigcup_{n=1}^\infty \left(r_n - \frac{1}{2^{m+n}}, r_n + \frac{1}{2^{m+n}}\right).$$

(i) Is it true that $A = \mathbb{Q}$? Justify your answer.
(ii) Find the measure of this set.

4. (10+10 pts.) Let f be a function on $[a,b]$ of total variation $T_a^b f < \infty$.
(i) Prove that $\displaystyle\int_{[a,b]} |f'| \leqslant T_a^b f$.
(ii) Prove that if f is absolutely continuous then equality holds in (i).

5. (10+10 pts.) Let $1 \leqslant r < p < s < \infty$.
(i) Show that

$$L^r[0,\infty) \cap L^s[0,\infty) \subset L^p[0,\infty).$$

(ii) Show that

$$L^s[0,1] \subset L^r[0,1].$$

Is it true that $L^s[1,\infty) \subset L^r[1,\infty)$? Justify your answer.

6. Let $\mathcal{H}_0$ be a closed linear subspace of the Hilbert space $\mathcal{H} = L^2[0,1]$ and let $f_0 \in \mathcal{H}$. Prove the equality

$$\min_{f\in\mathcal{H}_0} \|f_0 - f\| = \max_{\substack{g\in\mathcal{H}_0^\perp \\ \|g\|=1}} |\langle f_0, g\rangle|.$$