Math 540 Comprehensive Examination

August 22, 2006

Solve all problems from the first part and two problems from the second part. Indicate your choice. All problems have equal value.

m denotes the Lebesgue measure on $\mathbb{R}$.

Part I

1. Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function. Show directly that f is (Lebesgue) measurable and Lebesgue integrable, and that

$$(\mathscr{R})\int_a^b f(x)\,dx = (\mathscr{L})\int_{[a,b]} f\,dm.$$

2. Decide whether each of the following three statements is true or false. Justify your answer.

(a) If f has bounded variation on $[a,b]$, then f is Riemann integrable on $[a,b]$.

(b) $f(x) = \sqrt{x}\,\sin x$ is absolutely continuous on $[0,2\pi]$.

(c) Suppose $f_n : [0,1] \to \mathbb{R}$ are absolutely continuous functions with $T_0^1(f_n) \leq 1$ for every n and $f_n \to f$ uniformly on $[0,1]$. Does it follow that f is absolutely continuous on $[0,1]$?

3. For every $f \in L^\infty[0,1]$ show that:

(a) $\|f\|_p \leq \|f\|_{p'}$ whenever $1 \leq p < p' \leq \infty$.

(b) $\lim\limits_{p\to\infty} \|f\|_p = \|f\|_\infty$.

Part II

4. (a) State the Baire category theorem.

(b) Suppose $(f_\alpha)_{\alpha\in A}$ is a family of continuous functions on $\mathbb{R}$ such that $\sup\limits_{\alpha\in A}|f_\alpha(x)| < \infty$ for every $x \in \mathbb{R}$. Prove that there exists a non-empty open interval I such that

$$\sup_{x\in I}\sup_{\alpha\in A}|f_\alpha(x)| < \infty.$$

5. Let

$$f(x) = \int_0^\infty e^{-xy}\left(\frac{\sin y}{y}\right)^4 dm(y).$$

(a) Is f continuous on $[0,\infty)$?

(b) Is f differentiable on $(0,\infty)$?

Justify your answers.

6. Suppose $f : [0,1] \to [0,\infty)$ is measurable and satisfies

$$m\big(\{x \in [0,1] : f(x) > y\}\big) \leq \frac{1}{y(\ln y)^2} \quad \text{for every } y > 2.$$

Show that f is integrable.