Math 531 Comprehensive Exam
May 2009

Problem 1

(a) Let $f(n) = \sum_{p|n} \frac{1}{p}$. Prove an asymptotic for $\sum_{n \le x} f(n)$.

(b) Let g be the multiplicative function satisfying $g(p^a) = p^{a-1}(p+1)$ for prime p and $a \ge 1$. Prove a formula for $\sum_{n \le x} g(n)$ with error $O(x \log x)$. Express the constant in terms of values of the Riemann zeta function.

Problem 2

For this problem, any form of the prime number theorem may be used.

(a) When x is very large, determine which function is larger,

$$A(x) = x \sum_{p \le x} 1 \qquad \text{or} \qquad B(x) = \frac{3}{5} \sum_{x < p \le 2x} p.$$

(b) Determine asymptotically how many positive integers $\le x$ are odd, squarefree and have an even number of prime factors.

Problem 3

(i) Suppose t_n are complex numbers and $T(x) = \sum_{n \le x} t_n$ satisfies $T(x) = O(x^a)$, where $a \ge 0$. Prove that the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{t_n}{n^s}$$

represents an analytic function in the half-plane $\mathrm{Re}\, s > a$.

(ii) Let $t_n = \Lambda(n) - 1$. Give a **plausible** estimate for $T(x)$ that implies the Riemann Hypothesis, and prove the implication. By plausible, we mean an estimate which is not known to be false, such as $T(x) = O(1)$. In other words, use the weakest estimate for $T(x)$ that still implies RH.

Problem 4

Let χ be a nonprincipal Dirichlet character modulo q and let

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \qquad (\mathrm{Re}\, s > 1).$$

(i) Explain why $L(s,\chi)$ has an analytic continuation to the half-plane $\operatorname{Re} s > 0$.

(ii) Let

$$H(s) = \frac{\zeta(s)}{\phi(q)} \sum_{\chi \bmod q} L(s,\chi) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}.$$

Find the smallest number n (as a function of q) for which $h(n) \neq 1$.