Math 531 Comprehensive Exam
May 2008

Problem 1

Let $\Psi(x, y)$ be the number of positive integers $n \leq x$, all of whose prime factors are $\leq y$.

(i) Show, for $y \geq \sqrt{x}$, that

$$\Psi(x, y) = \lfloor x \rfloor - \sum_{y < p \leq x} \left\lfloor \frac{x}{p} \right\rfloor,$$

where $\lfloor z \rfloor$ is the greatest integer $\leq z$.

(ii) Without using the Prime Number Theorem (you may use the Chebyshev and Mertens estimates), find an approximate formula for $\Psi(x, y)$, for $\sqrt{x} \leq y \leq x$, in terms of elementary functions of x and y and with an error term no worse than $O(x/\log x)$.

Problem 2

(i) Describe precisely a number-theoretic function $f(n)$ such that

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \qquad (\mathrm{Re}\, s > 1).$$

Compute $f(35)$, $f(105)$, $f(315)$.

(ii) Let $s = \sigma + it$, where $\sigma > 1$ and t is real. Prove that

$$\left| \frac{1}{\zeta(s)} \right| \leq \frac{\zeta(\sigma)}{\zeta(2\sigma)}.$$

(iii) State a formula, without proof, for $\sum_{n \leq x} f(n)$ as an integral in the complex plane involving $\zeta(s)/\zeta(2s)$.

Problem 3

Define $\pi(x)$ to be the number of primes $\leq x$, p_n the n-th prime number, $\psi(x) = \sum_{n \leq x} \Lambda(n)$ and $M(x) = \sum_{n \leq x} \mu(n)$, where Λ is the von Mangoldt function and μ is the Möbius function.

(i) Prove that $\Lambda(n) = \sum_{d|n} \mu(d) \log(n/d)$

(ii) Give, without proof, statements about $\pi(x)$, p_n, $\psi(x)$ and $M(x)$ which are equivalent to the Prime Number Theorem.

(iii) State, without proof, a result about the zeros of $\zeta(s)$ which is needed in the analytic proof of the Prime Number Theorem.

(iv) State the Riemann Hypothesis about $\zeta(s)$.

(v) Give a statement about **one** of $\pi(x)$, p_n, $\psi(x)$, $M(x)$ that is equivalent to the Riemann Hypothesis.

Problem 4

Without using the Prime Number Theorem, prove the equivalence of the statements given in Problem 3 (ii) for $\pi(x)$ and $\psi(x)$.

Problem 5

Let χ be a Dirichlet character mod q, and put $L(s, \chi) = \sum_{n \leq x} \chi(n) n^{-s}$.

(i) Evaluate

$$f(\chi) = \lim_{s \to 1} \frac{L(s, \chi)}{\zeta(s)}.$$

Make sure your answer covers all possibilities for χ.

(ii) Show how

$$\sum_{\chi \bmod q} \log L(s, \chi)$$

can be written as a multiple sum that **does not** involve Dirichlet characters.