Math 531 Comprehensive Exam
May 2007

1. *Without using the Prime Number Theorem* (you may use Chebychev and/or Mertens type estimates), determine whether the series $\sum_{p\geq 3}(p\log\log p)^{-1}$ converges or diverges. If it converges, obtain an estimate for the tails $R(x) = \sum_{p>x}(p\log\log p)^{-1}$; if it diverges, obtain an asymptotic formula for the partial sums $S(x) = \sum_{n\leq x}(p\log\log p)^{-1}$.

2. Let $S(x) = \sum_{p\leq x}\log(x/p)$. Obtain an asymptotic estimate for $S(x)$ with an error term that is by a factor $O(1/\log x)$ smaller than the main term. (You may use any standard version of the Prime Number Theorem.)

3. Let $d_{even}(n)$, resp. $d_{odd}(n)$, denote the number of even, resp. odd, (positive) divisors of n, and let $f(n) = d_{even}(n) - d_{odd}(n)$.

(i) Evaluate the Dirichlet series $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$, and express it in terms of the Riemann zeta function.

(ii) Show that $\sum_{n\leq x} f(n) = o\left(\sum_{n\leq x} d(n)\right)$ as $x\to\infty$.

4. Show that there exists a positive constant c such that for all positive integers n,

$$\sum_{d|n} d \geq c\frac{n^2}{\phi(n)},$$

and determine, with proof, the *best-possible* value for this constant. In other words, compute

$$L = \inf_{n\in\mathbf{N}} \frac{\phi(n)}{n^2}\sum_{d|n} d.$$

(Note that L should be given as a simple *explicit* expression involving familiar constants, and not, for example, sums or products over primes.)

5. Let

$$F(s) = \sum_{n=1}^{\infty}\frac{\{n/3\}\log n}{n^s},$$

where $\{t\} = t - [t]$ is the fractional part of t.

(i) Express $F(s)$ in terms of the Riemann zeta function and/or Dirichlet L-series.

(ii) Show that $F(s)$ has a meromorphic continuation to the half-plane $\sigma > 0$ and determine all its poles (if any) in this half-plane, *along with their order.* (I.e., determine for each pole whether it is simple, double, etc.; there is no need to compute residues.)