Math 531 Comprehensive Exam
May 2006

1. Let $f(n)$ be the arithmetic function defined by the Dirichlet series

$$F(s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\frac{\zeta(s)}{\zeta(3s)}\quad(\sigma>1),$$

where $\zeta(s)$ is the Riemann zeta function.

(i) Express $F(s)$ as an Euler product, and determine the values of f at prime powers.

(ii) Estimate $\sum_{n\le x}f(n)$ with as good an error as you can get.

2. Suppose χ is a non-principal character modulo q. By the general theory of L-series the Dirichlet L-series $L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s}$ converges at $s=1$. Obtain an *explicit* bound (as good as you can get) in terms of x and q for the tails of this series, $R(x,\chi)=\sum_{n>x}\chi(n)n^{-1}$. (Explicit here means that the bound should involve concrete numerical constants, rather than O's or unspecified c's.)

3. Obtain an asymptotic estimate for the sum

$$S(x)=\sum_{x<p\le 2x}\frac{1}{p}.$$

with *relative* error $1/\log x$ (i.e., an estimate of the form $S(x)=f(x)(1+O(1/\log x))$ with a simple elementary function $f(x)$. (You may use any standard prime number estimates including the estimates of Chebychev, Mertens, and the Prime Number Theorem with error term.)

4. For $x\ge 2$, let

$$Q(x)=\prod_{p\le x}\left(1+\frac{2}{p}\right).$$

Without using the Prime Number Theorem (you may use Chebychev and/or Mertens type estimates), obtain an asymptotic formula for $Q(x)$, i.e., find a simple elementary function $f(x)$ such that $Q(x)\sim f(x)$. (You can leave constants in this estimate unspecified. Partial credit will be given if you can only prove that $f(x)$ represents the correct order of magnitude of $Q(x)$, i.e., for a proof that $Q(x)\ll f(x)$ and $Q(x)\gg f(x)$ for $x\ge 2$.)

5. Let

$$F(s)=\frac{\zeta(s)}{\zeta(3s)},$$

where $\zeta(s)$ is the Riemann zeta function.

(i) Determine all poles of $F(s)$ and their residues in the half-plane $\sigma>1/3$.

(ii) Assume the Riemann Hypothesis. Describe as precisely as you can the location of the zeros and poles of $F(s)$ in the strip $0<\sigma<1$.