Math 531 Comprehensive Exam
January 2009

Problem 1

(i) State, without proof, an asymptotic formula for $\psi(x) = \sum_{n \le x} \Lambda(n)$.

(ii) State, without proof, an estimate for $M(x) = \sum_{n \le x} \mu(n)$ which is equivalent to the prime number theorem.

(iii) Find, with proof, an asymptotic formula for $\sum_{p \le x} \frac{\sqrt{\log p}}{p}$.

Problem 2

Let $a > 0$. Define an arithmetic function k_a by

$$\zeta(s)\zeta(s-a) = \sum_{m=1}^{\infty} \frac{k_a(m)}{m^s} \qquad (\operatorname{Re} s > a+1).$$

Here ζ is the Riemann zeta function.

(i) Give a combinatorial or number-theoretic interpretation for $k_a(m)$.

(ii) Prove that k_a is a multiplicative function.

(iii) Let $a > 1$. Find an asymptotic formula for

$$\sum_{m \le x} k_a(m)$$

with *relative* error $O(1/x)$ (that is, show that the sum is $f(x)(1 + O(1/x))$ for some "simple" function f).

Problem 3

(i) Give an infinite product expansion for the Gamma function $\Gamma(z)$. State the location of all zeros and poles of $\Gamma(z)$.

(ii) State the functional equation relating $\zeta(s)$ to $\zeta(1-s)$.

(iii) Let $\operatorname{Re} s > 1$. Express

$$I(s) = s \int_0^{\infty} e^{-st} \psi(e^t) dt$$

in terms of $\zeta(s)$.

(iv) State, as precisely as possible, the location of the poles of the meromorphic continuation of $I(s)$ to the complex plane. Be sure to say whether $I(s)$ has finitely many or infinitely many poles.

Problem 4

(i) Show that there are infinitely many primes whose last four digits are 2009.

(ii) Let $q > 1$. Determine the limit

$$\lim_{s\to 1^+} \sum_{n\equiv 1 \pmod q} \frac{\mu(n)}{n^s}.$$

Be sure to justify why the limit exists.