# Math 531 Comprehensive Exam
# January 2008

## Problem 1

(a) Find an asymptotic formula for the sum $\sum_{p \le x} \frac{\log^2 p}{p^{1/2}}$, where the sum is over primes $p$.

(b) Let $[a_1, a_2, \dots, a_n]$ be the least common multiple of $a_1, a_2, \dots, a_n$. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{z^n}{[1, 2, \dots, n]}.$$

## Problem 2

Suppose $S(x) = \sum_{n \le x} a_n \sim Lx$, where $a_1, a_2, \dots$ are real numbers and $L$ is a nonzero real number. Consider the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

(a) Show that $F(s)$ has abscissa of convergence $\sigma_c = 1$.

(b) Find an asymptotic formula for $\sum_{n \le x} \frac{a_n}{n}$.

## Problem 3

(a) Prove that $\tau(n^2)$ is multiplicative, where $\tau(m)$ is the number of positive divisors of $m$.

(b) Prove that

$$\sum_{n=1}^{\infty} \frac{\tau(n^2)}{n^s} = \frac{\zeta^3(s)}{\zeta(2s)} \qquad (\operatorname{Re} s > 1).$$

Hint: use the identity $\sum_{m=0}^{\infty} (2m+1)x^m = \frac{1+x}{(1-x)^2}$ for $|x| < 1$.

(c) Assuming the Riemann Hypothesis, describe as accurately as possible the location of the poles of the right side of the equation in (b).

**Problem 4**

(a) Use the multiplicative properties of the Möbius function $\mu$ to show that

$$\sum_{d^2|n} \mu(d) = \begin{cases} 1 & n \text{ is squarefree} \\ 0 & \text{otherwise} \end{cases}.$$

(b) If $n$ has prime factorization $n = p_1^{e_1} \cdots p_r^{e_r}$, let $\lambda(n) = (-1)^{e_1+\cdots+e_r}$. Find an asymptotic formula for

$$\sum_{\substack{n \le x \\ \lambda(n) \ne \mu(n)}} 1.$$

**Problem 5**

Let $\chi$ be a *real*, nonprincipal character modulo $k$.

(a) Show that there are infinitely many *primes* $q$ with $\chi(q) = 1$ and infinitely many *primes* $q$ with $\chi(q) = -1$.

(b) Determine whether or not the series $\sum_{n=1}^{\infty} \frac{\chi(n)}{\sqrt{n}}$ converges or diverges.