Math 531 Comprehensive Exam
January 2007

1. Determine which of the three functions

$$\frac{x}{\log x}, \quad \frac{x}{\log x - 1}, \quad \frac{x}{\log x + 1},$$

represents, for large x, the best approximation to the prime counting function $\pi(x)$. Justify your answer with appropriate estimates. (You may use any standard form of the Prime Number Theorem.)

2. Let

$$S(x) = \sum_{n \le x} \frac{\phi(n!)}{n!}.$$

Obtain an asymptotic estimate for $S(x)$, consisting of a simple function of x as main term and an error term that is by a factor $O(1/\log x)$ smaller than the main term. (Note that the summands in $S(x)$ involve a factorial, $n!$, in both numerator and denominator.)

3. Obtain an asymptotic estimate for the sum

$$S(x) = \sum_{ab \le x} \frac{a}{b},$$

with error term $O(x)$. (The summation here is over all pairs (a, b) of positive integers (not necessarily coprime) satisfying $ab \le x$.)

4. Let $f(n)$ be the number of ordered triples (a, b, c) of positive integers with $n = ab^2c^3$. (For example, $f(8) = 3$ since 8 has three representations of the form ab^2c^3: $1 \cdot 1^2 \cdot 2^3$, $2 \cdot 2^2 \cdot 1^3$, and $8 \cdot 1^2 \cdot 1^3$.)

(i) Express the Dirichlet series $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ in terms of the Riemann zeta function.

(ii) State a formula for the partial sums $M(x) = \sum_{n \le x} f(n)$ as a complex integral involving the Dirichlet series $F(s)$. (No proofs required for this part, but state any hypotheses that are needed for your formula to be valid. You may assume that x is not an integer.)

(iii) *Using this representation of $M(x)$ as a complex integral* "guess" an asymptotic formula for $M(x)$ of the form $M(x) \sim Ax$, with an appropriate constant A. (Again no proofs are required, but you should say how you arrived at your guess, and, in particular, determine the constant A in terms of the Riemann zeta function.)

5. Let $f(n)$ be the arithmetic function defined by

$$f(n) = \begin{cases} \mu(n) & \text{if } n \equiv 1 \bmod 3, \\ -\mu(n) & \text{if } n \equiv 2 \bmod 3, \\ 0 & \text{if } n \equiv 0 \bmod 3. \end{cases}$$

and let $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ be the associated Dirichlet series.

(i) Does $F(s)$ have an Euler product representation in the half-plane $\sigma > 1$? If so, find the representation, i.e., compute the factors $F_p(s)$ arising in this representation. If not, state why an Euler product for $F(s)$ does not exist.

(ii) Citing appropriate theorems, prove that $F(s)$ has a **meromorphic** continuation to the half-plane $\sigma > 0$, and find all **zeros** of $F(s)$ (if any) in this half-plane.

(iii) Does $F(s)$ has a **pole** at $s = 1$? Explain, citing appropriate theorems if needed.