# Math 531 Comprehensive Exam
# January 2006

**1.** Let $f(n) = \sum_{d|n} (\log d)/d$, and let $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$.

(i) Evaluate $F(s)$ in terms of the Riemann zeta function.

(ii) Obtain an estimate for the sums $S(x) = \sum_{n \le x} f(n)$ with error term $O((\log x)^2)$. (This estimate will involve a constant; for full credit express this constant in terms of the Riemann zeta function.)

**2.** *Without using the Prime Number Theorem* (you may use Chebychev and/or Mertens type estimates), determine whether the series $\sum_{n=2}^{\infty} \Lambda(n)/(n \log n)$ converges, and obtain an estimate for the partial sums

$$S(x) = \sum_{2 \le n \le x} \frac{\Lambda(n)}{n \log n},$$

with as good an error term you can get without appealing to the PNT. (Here $\Lambda(n)$ denotes the von Mangoldt function.)

**3.** Let $f(n)$ be an arithmetic function satisfying

$$S(x) = \sum_{n \le x} f(n) = Ax^{\alpha} + Bx^{\beta} + O(x^{\delta}) \quad (x \ge 1),$$

where $\alpha > \beta > \delta \ge 0$ are real numbers and $A$ and $B$ are *non-zero* real numbers. Let $F(s) = \sum_{n \ge 1} f(n) n^{-s}$ be the generating Dirichlet series for $f$. Find, with proof, a half-plane (as large as possible) in which $F(s)$ is guaranteed to have a meromorphic continuation, and determine all poles (if any) of (the meromorphic continuation of) $F(s)$ in that region, and the residues of $F$ at those poles.

**4.** Let $f(n)$ denote the remainder of $n$ modulo 5 (so that $f(n) \in \{0, 1, 2, 3, 4\}$ for each $n$), and let $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$ be the Dirichlet series of $f$.

(i) Express $F(s)$ in the half-plane $\sigma > 1$ in terms of Dirichlet $L$-functions.

(ii) Show that $F(s)$ has a merophorphic continuation to the half-plane $\sigma > 0$ and find all poles (if any) of $F(s)$ in this half-plane, and their residues.

**5.** Let $Q_+(x)$ (resp. $Q_-(x)$) denote the number of squarefree positive integers $\le x$ that have an even (resp. odd) number of prime factors.

(i) State an estimate involving $Q_+(x)$ and $Q_-(x)$ that is equivalent to the Prime Number Theorem. (No proof required; one or two lines of explanation, citing appropriate theorems, are sufficient.)

(ii) Is it possible that $Q_{\pm}(x)$ satisfy the estimate $Q_+(x) = Q_-(x) + O(1)$? Explain why, or why not, citing appropriate theorems.