Math 531 Comprehensive Exam
August 2007

Problem 1

Show that $\sum_{n\leq x}\left\{\frac{x}{n}\right\} = Cx + O(x^{1/2})$ and identify the constant C. You may use the fact that $\sum_{n\leq x}\tau(n) = \sum_{d\leq x}\lfloor x/d\rfloor$ (here $\tau(n)$ is the number of positive divisors of n, and $\{x\} = x - \lfloor x\rfloor$ is the fractional part of x).

Problem 2

(a) Let $A(x)$ be the number of primes in $[\sqrt{x}, x]$ and let $B(x)$ be the number of primes in $(x, 4x]$. Find $\lim_{x\to\infty} B(x)/A(x)$. You may use the Prime Number Theorem.

(b) Find the **supremum** of real numbers r for which

$$\sum_{e^m\leq p\leq e^{m+1}} \frac{1}{p} \ll \frac{1}{m^r}, \qquad (m\geq 1).$$

Do **not** use the Prime Number Theorem, but only the Chebyshev and/or Mertens estimates.

Problem 3

Suppose f is an arithmetic function defined by $n = \sum_{d|n} f(d)\mu(n/d)$.

(a) Express f in terms of familiar arithmetic functions.

(b) Express $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ in terms of the Riemann zeta function.

Problem 4

Let χ be a real, nonprincipal Dirichlet character, and for $\mathrm{Re}\, s > 1$ set

$$F(s) = L(s,\chi)\zeta(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

(a) Can $a_{100} = 0$? Explain.

(b) Show that $F(s)$ is meromorphic in the entire complex plane and determine all poles (if any) of $F(s)$ along with associated residues.

Problem 5

Let $M(x) = \sum_{n\leq x}\mu(n)$.

(a) Express $1/\zeta(s)$ in terms of an integral involving $M(x)$ (with proof). Be sure to specify for which values of s the formula is valid.

(b) For $x > 1$ and x not an integer, express $M(x)$ in terms of an integral involving $\zeta(s)$. No proof required.

(c) Show that if $M(x) = O(x^a)$ for some real number $a > 0$, then $\zeta(s)$ has no zeros in the half-plane $\mathrm{Re}\, s > a$.