Math 531 Comprehensive Exam
August 2006

1. Let $f(n)$ denote the number of representations of n as a product of two coprime positive integers, i.e.,

$$f(n) = \{(a,b) \in \mathbb{N}^2 : ab = n,\ (a,b) = 1\}.$$

(i) Define g by the identity $f = g * d$, where d is the divisor function. Determine the function g explicitly, i.e., describe the values $g(n)$ explicitly, in terms of a well-known arithmetic function (not as a sum).

(ii) Let $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$. Express $F(s)$ (a) in terms of the Riemann zeta function and (b) as an Euler product.

2. (i) *Using only Chebyshev estimates (but not the Prime Number Theorem)*, show that there are infinitely many primes that contain all 10 digits $0, 1, \dots, 9$ in their decimal representation.

(ii) Show that there are infinitely many primes whose decimal representation *begins* with the digits 9876543210. (For this part you may use the Prime Number Theorem.)

3. Let f be an arithmetic function satisfying

$$(*) \qquad \left| \sum_{n \le x} f(n) \right| \le x^{1/3}, \quad x \ge 1$$

and let $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ be the associated Dirichlet series.

(i) Determine a half-plane (as large as possible, given the assumption $(*)$) in which $F(s)$ converges.

(ii) Obtain an *explicit* bound (as good as you can get) for $|F(s)|$ in this half-plane in terms of $\sigma = \Re s$ and $t = \Im s$. (Explicit here means that the bound should be an explicit function of σ and t (such as $|F(s)| \le 10e^{|t|}\sigma$), and not involve O's or unspecified constants.)

4. (i) Obtain an estimate for the sum

$$S(x) = \sum_{\substack{n \le x \\ n \text{ odd}}} \frac{1}{n}, \qquad x \ge 1,$$

with error term $O(1/x)$. (The estimate should not involve any unspecified constants.)

(ii) Let

$$D(x) = \sum_{\substack{n \le x \\ n \text{ odd}}} d(n),$$

where $d(n)$ is the divisor function. Give an estimate for $D(x)$ with error term $O(\sqrt{x})$. As in (i), any constants arising in this estimate should be worked out explicitly. (Hint: Use Dirichlet's hyperbola method and the result of part (i).)

5. Let

$$F(s) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n/3)}{n^s}.$$

(i) Express $F(s)$ in terms of the Riemann zeta function and/or Dirichlet L-series.

(ii) Determine, with proof, the abscissa of absolute convergence σ_a and the abscissa of convergence σ_c of $F(s)$.

(iii) Show that $F(s)$ has a meromorphic continuation to the half-plane $\sigma > 0$ and determine all its poles (if any) in this half-plane.