MATH 530 - Comprehensive Examination, May 2018

Instructions: Attempt all problems. Each question is worth 20 points.

Here is a theorem that you may wish to quote in your solutions:

Minkowski's bound: Let $K/\mathbb{Q}$ be a finite extension of the rational numbers with degree n and let O_K be the set of algebraic integers in K and Δ the discriminant of O_K over the integers. Assume K has r embeddings into the real numbers and $2s$ embeddings into the complex numbers. Then every class of fractional ideals contains an ideal I in O_K that satisfies

$$|N_{K/\mathbb{Q}}(I)| \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s |\Delta|^{1/2}.$$

1. Determine the structure of the ideal class group of $\mathbb{Q}(\sqrt{35})$.

2. Let $K = \mathbb{Q}(\zeta_{21})$, where $\zeta_{21} = e^{2\pi i/21}$ is a primitive 21-st root of unity.

(a) Give the degree $[K : \mathbb{Q}]$ of the extension $K/\mathbb{Q}$.

(b) For each of the primes p in $\mathbb{Z}$ that ramify in $K/\mathbb{Q}$ determine the number of distinct primes P in $\mathbb{Z}[\zeta_{21}]$ that divide p.

(c) For each of the primes $p = 2$ and $p = 13$ in $\mathbb{Z}$ determine the number of distinct primes P in $\mathbb{Z}[\zeta_{21}]$ that divide p.

3. Let K be a number field and let O_K be its ring of integers. Prove that if $\alpha \in O_K$ satisfies $\log|\sigma(\alpha)| = 0$ for every field embedding $\sigma : F \hookrightarrow \mathbb{C}$, then α is a root of unity.

4. Let O_K be a Dedekind domain with fraction field K and let $Q \subseteq O_K$ be a nonzero prime ideal. Let L/K be a finite Galois extension and let O_L be the integral closure of O_K in L.

(a) Prove that $\mathrm{Gal}(L/K)$ acts transitively on the set of prime ideals $P \subset O_L$ above Q.

(b) Prove that if P_1 and P_2 are two primes ideals in O_L above Q, then their decomposition groups D_{P_1} and D_{P_2} are conjugate.

5. (a) Give an example of $d \in \mathbb{Z}$ such that d is a square in $\mathbb{Q}_3$ and in $\mathbb{Q}_5$, but not in $\mathbb{Q}_7$.

(b) For your example d from part (a), and for each of $p = 3, 5, 7$, how many ways are there to extend the p-adic absolute value on $\mathbb{Q}$ to $\mathbb{Q}(\sqrt{d})$?