MATH 530 - Comprehensive Examination, May 2009

Instructions: Do any four of these five problems. Indicate which problem you have omitted. Each question is worth 20 points.

Here is a theorem that you may wish to quote in your solutions:

Minkowski's bound: Let $K/\mathbb{Q}$ be a finite extension of the rational numbers with degree n and let O_K be the set of algebraic integers in K and Δ the discriminant of O_K over the integers. Assume K has r embeddings into the real numbers and $2s$ embeddings into the complex numbers. Then every class of fractional ideals contains an ideal I in O_K that satisfies

$$|N_{K/\mathbb{Q}}(I)| \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s |\Delta|^{1/2}.$$

1. Let $K = \mathbb{Q}(\alpha)$, with α a root of $x^3 + x^2 + x - 2$. You may use the fact that α has discriminant -139.

a) Find an integral basis for the ring of integers O_K of K.

b) Factor the ideals $2O_K$ and $3O_K$ into prime ideals.

c) Show that O_K is a principal ideal domain.

2. Let $K = \mathbb{Q}(\zeta_7, \sqrt{2})$, with $\zeta_7 = e^{2\pi i/7}$. Let O_K be the ring of integers of K and let $P|2$ and $Q|5$ be prime ideals in O_K that lie over the rational primes $p = 2$ and $p = 5$, respectively.

a) Find the decomposition and inertia fields associated to $P|2$.

b) Find the decomposition and inertia fields associated to $Q|5$.

c) Suppose that $p \in \mathbb{Z}$ is a rational prime. Show that the ideal pO_K is not a prime ideal of O_K.

3. Let $K = \mathbb{Q}(\alpha)$, with α a root of $x^3 + x^2 + x - 2$.

a) State Dirichlet's Unit Theorem.

b) Describe the structure of the unit group E_K of the ring of integers of K as an abstract group.

c) Explicitly find a subgroup of finite index in the group of units E_K.

4. Let K be a number field with ring of integers O_K. Let L be a finite galois extension of K with group $G = \mathrm{Gal}(L/K)$ and with ring of integers O_L.

a) If P is a prime ideal of O_L and $\sigma \in G$ show that $\sigma(P)$ is a prime ideal of O_L, and $P \cap K = \sigma(P) \cap K$.

b) Let P and P' be prime ideals of O_L above the same prime ideal of O_K. Prove that there exists $\sigma \in G$ such that $\sigma(P) = P'$.

5. Let $\Phi_{10}(x) \in \mathbb{Z}[x]$ be the 10th cyclotomic polynomial.

a) Find $\Phi_{10}(x)$ explicitly.

b) Describe the factorization of $\Phi_{10}(x)$ in $\mathbb{Q}_{11}[X]$ (where $\mathbb{Q}_{11}$ denotes the complete field of 11−adic numbers). Give the number of irreducible factors and their degrees.