MATH 530 - Comprehensive Examination - May 2006

Instructions: Do four of the following six problems. Select two problems from $1, 2, 3$ and two problems from $4, 5, 6$. Indicate clearly which problems you have selected. Each question is worth 25 points, Maximum score is 100 points.

Here is a theorem that you may wish to quote in your solutions:

Minkowski's bound: Let K be a finite extension of the rational numbers with degree n and let O_K be the set of algebraic integers in K and Δ the discriminant of O_K over the integers. Assume K has r embeddings into the real numbers and $2s$ embeddings into the complex numbers. Then every class of fractional ideals contains an ideal I in O_K that satisfies

$$|N_{K/\mathbb{Q}}(I)| \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s |\Delta|^{1/2}.$$

***** Select two problems from $1, 2, 3$ *****

1) Let $K = \mathbb{Q}(\sqrt{61})$. Show that the ring of integers O_K of K is a principal ideal domain.

2) Let K be a number field with ring of integers O_K and group of units E_K.

a) State Dirichlet's Unit Theorem for E_K.

b) Let $K = \mathbb{Q}(\sqrt[3]{2})$ and let $L/\mathbb{Q}$ be the normal closure of $K/\mathbb{Q}$. Describe E_K and E_L as abstract abelian groups.

c) Find subgroups of finite index in E_K and E_L with explicit generators.

3) Let $K = \mathbb{Q}(\sqrt{21}, \sqrt{33})$.

a) For each of the primes $p = 2, 3$ give the decomposition parameters e, f, r for the decomposition of p in $K/\mathbb{Q}$.

b) For each of the primes $p = 2, 3$ give the decomposition field and the inertia field for $K/\mathbb{Q}$.

***** Select two problems from 4, 5, 6 *****

4) Let K be a number field with ring of integers O_K. Given any ideal class c of K and any ideal I of O_K, show that there exists an ideal J of O_K in the class c that is relatively prime to I.

5) Let K be a number field with ring of integers $O_K = \mathbb{Z}[\alpha]$. Let f be the minimum polynomial for α over $\mathbb{Q}$. For a prime p in $\mathbb{Z}$ let

$$\bar{f} = \bar{f}_1 \cdot \bar{f}_2 \cdots \bar{f}_r \pmod p$$

be the factorization of f modulo p into irreducible factors. Denote by I_1 the ideal $(p, f_1(\alpha)) \subset \mathbb{Z}[\alpha]$, where $f_1 \in \mathbb{Z}[x]$ is a polynomial that reduces modulo p to $\bar{f}_1$. Assume that $I_1 \neq \mathbb{Z}[\alpha]$.

Show that I_1 is a prime ideal of O_K dividing p.

6) Show that the quotient group $\mathbb{Q}_5^*/(\mathbb{Q}_5^*)^2$ is finite, where Q_5 is the complete 5−adic field.