MATH 530 - Comprehensive Examination, January 2012

Instructions: Do any four of these five problems. Indicate which problem you have omitted. Each question is worth 20 points.

Here is a theorem that you may wish to quote in your solutions:

Minkowski's bound: Let $K/\mathbb{Q}$ be a finite extension of the rational numbers with degree n and let O_K be the set of algebraic integers in K and Δ the discriminant of O_K over the integers. Assume K has r embeddings into the real numbers and $2s$ embeddings into the complex numbers. Then every class of fractional ideals contains an ideal I in O_K that satisfies

$$|N_{K/\mathbb{Q}}(I)| \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s |\Delta|^{1/2}.$$

1. Determine the structure of the ideal class group of $\mathbb{Q}(\sqrt{-37})$.

2. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ and let O_K be the ring of integers of K.

a) For a prime P of O_K that divides the rational prime p define the decompositon field and the inertia field.

b) Let $P|p$ and $P'|p$ be distinct primes of O_K that divide the same rational prime p. Show that the decomposition fields and the inertia fields are the same for $P|p$ and for $P'|p$.

c) Find the decomposition field and the inertia field associated to a prime $P|7$.

d) Find the decomposition field and the inertia field associated to a prime $P|3$.

3. Let $K = \mathbb{Q}(\zeta)$, with $\zeta = e^{2\pi i/24}$ a primitive 24-th root of unity.

a) State Dirichlet's Unit Theorem.

b) Describe the structure of the unit group E_K of the ring of integers of K as an abstract group.

c) Explicitly describe the real quadratic subfields of K.

d) Explicitly find a subgroup of finite index in the group of units E_K.

4. Let K be a number field with ring of integers O_K.

a) Prove that there exists a constant $C > 0$ (depending on K) such that every nonzero ideal I of O_K contains a nonzero element α with $|N_{K/\mathbb{Q}}(\alpha)| \leq C\,|N_{K/\mathbb{Q}}(I)|$.

b) Deduce from a) that the class number of O_K is finite.

5. Let f be the polynomial $x^4 + 4x^2 + 10 \in \mathbb{Q}[x]$.

a) Show that f is irreducible over $\mathbb{Q}$.

b) Find the degree of each irreducible factor of f in the polynomial ring $\mathbb{Q}_7[X]$. Here $\mathbb{Q}_7$ denotes the complete field of 7−adic numbers.