Math 530 Comprehensive Exam. January 2008

Instructions: Do four of the following five problems.

Here is a theorem that you may wish to quote in your solutions:

Minkowski's bound: Let K be a finite extension of the rational numbers with degree n and let O_K be the set of algebraic integers in K and Δ the discriminant of O_K over the integers. Assume K has r embeddings into the real numbers and $2s$ embeddings into the complex numbers. Then every class of fractional ideals contains an ideal I in O_K that satisfies

$$|N_{K/\mathbb{Q}}(I)| \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s |\Delta|^{1/2}.$$

Problem 1

Determine the structure of the ideal class group of $\mathbb{Q}(\sqrt{26})$, and explicity give ideals which represent the different ideal classes.

Problem 2

a) State Dirichlet's Unit Theorem, and explain what it says about the field $K = \mathbb{Q}(\sqrt{-3}, \sqrt{-5})$.
b) Determine a subgroup of finite index in the group of units of the ring of integers of K.

Problem 3

Let K be a number field which is Galois over $\mathbb{Q}$ and let $\mathcal{O}_K$ be its ring of integers.
a) Define the inertia subgroup I_P for a prime ideal P of $\mathcal{O}_K$.
b) Prove that $\mathrm{Gal}(K/\mathbb{Q})$ is generated by the groups I_P as P runs over all primes of $\mathcal{O}_K$. Hint: consider the field fixed by the group generated by the I_P.

Problem 4

Let $L = \mathbb{Q}(\omega_{35})$, where $\omega_{35} = e^{\frac{2\pi i}{35}}$, and let $\mathcal{O}_L$ be the ring of integers of L.
a) Determine the structure of the Galois group $G = G(L/\mathbb{Q})$ of L over $\mathbb{Q}$.
b) Find all quadratic subfields of L over $\mathbb{Q}$.
c) Determine the ramification index e_p, the inertial degree f_p, and the number r_p of primes of $\mathcal{O}_L$ lying over p, for each of the rational primes $p = 2, 3, 7$.
d) Determine the decomposition field $L(p)$, for each of the rational primes $p = 2, 3, 7$. (Here $L(p) = L(q|p)$ is independent of the prime $q|p$ since G is abelian.)

Problem 5

Let $\mathbb{Q}_p$ denote the p-adic rational field (the completion of $\mathbb{Q}$ at a p-adic valuation) and let $(\mathbb{Q}_p^*)^3 = \{a^3 : a \in \mathbb{Q}_p^*\}$ be the subgroup of units that are perfect cubes.
a) State Hensel's lemma about reducibility of polynomials over Z_p.
b) Show that $6 \in (\mathbb{Q}_7^*)^3$.