MATH 530 - Comprehensive Examination, January 2007

Instructions: Do any four of these five problems. Indicate which problem you have omitted. Each question is worth 20 points.

Here is a theorem that you may wish to quote in your solutions:

Minkowski's bound: Let K be a finite extension of the rational numbers with degree n and let R be the set of algebraic integers in K and Δ the discriminant of R over the integers. Assume K has r embeddings into the real numbers and $2s$ embeddings into the complex numbers. Then every class of fractional ideals contains an ideal A in R that satisfies

$$|N_{K/\mathbb{Q}}(A)| \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s |\Delta|^{1/2}.$$

1. Let K be a finite Galois extension of $\mathbb{Q}$ with abelian Galois group G and ring of integers O_K.

a) If P is a prime ideal of O_K and $\sigma \in G$ show that $\sigma(P)$ is a prime ideal of O_K, and $P \cap \mathbb{Q} = \sigma(P) \cap \mathbb{Q}$.

b) Show that the primes P and $\sigma(P)$ have the same decomposition field.

c) Let $K = \mathbb{Q}(\zeta_{55})$, with $\zeta_{55} = e^{2\pi i/55}$. For each of the rational primes $p = 5, 11$, determine the invariants e, f, r that appear in the decomposition of pO_K as a product of prime ideals in O_K.

2. Let E_K be the unit group of the ring of integers of the number field K.

a) State Dirichlet's Unit Theorem for the structure of the unit group E_K.

b) For $K = \mathbb{Q}(\sqrt{-3}, \sqrt{-5})$, describe the structure of E_K as an abstract group.

c) For the group of units E_K under b), explicitly find a subgroup of finite index.

3. Let $K = \mathbb{Q}(\sqrt{-26})$ and let R be the ring of integers of K. Show that there exist ideals I and J in R such that $2R = I^2$ and $(1 + \sqrt{-26})R = J^3$. Determine the class number of K.

4. Prove that for a number field K with ring of integers O_K, every nonzero prime ideal of O_K is a maximal ideal. Hint: Show that for every ideal I there exists $m \in \mathbb{Z}$ such that $mO_K \subset I \subset O_K$ and use that O_K is a free $\mathbb{Z}$-module of finite rank.

5. Show that the polynomial $f = X^4 - 2X + 2$ is irreducible over $\mathbb{Q}$. Find the degree of each irreducible factor of f in the polynomial ring $\mathbb{Q}_5[X]$. Here $\mathbb{Q}_5$ denotes the complete field of 5-adic numbers.