MATH 530 - Comprehensive Examination, August 2009

Instructions: Do any four of these five problems. Indicate which problem you have omitted. Each question is worth 20 points.

Here is a theorem that you may wish to quote in your solutions:

Minkowski's bound: Let $K/\mathbb{Q}$ be a finite extension of the rational numbers with degree n and let O_K be the set of algebraic integers in K and Δ the discriminant of O_K over the integers. Assume K has r embeddings into the real numbers and $2s$ embeddings into the complex numbers. Then every class of fractional ideals contains an ideal I in O_K that satisfies

$$|N_{K/\mathbb{Q}}(I)| \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s |\Delta|^{1/2}.$$

1. Determine the structure of the ideal class group of $\mathbb{Q}(\sqrt{-35})$.

2. Let $K = \mathbb{Q}(\sqrt{5}, \sqrt{11})$ and let O_K be the ring of integers of K.

a) For each of the rational primes $p = 3, 5$, determine the invariants e, f, r that appear in the decomposition of pO_K as a product of prime ideals in O_K.

b) For each of the rational primes $p = 3, 5$, determine the decomposition field and the inertia field for $K/\mathbb{Q}$.

3. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

a) State Dirichlet's Unit Theorem.

b) Describe the structure of the unit group E_K of the ring of integers of K as an abstract group.

c) Explicitly find a subgroup of finite index in the group of units E_K.

4. Let K be a finite dimensional extension field of the rational numbers with $K \neq \mathbb{Q}$. Use the discriminant to show that there is a rational prime that ramifies in K.

5. Let $\mathbb{Q}_p$ denote the field of p-adic rational numbers and let $\mathbf{Z}_p$ be the subring of p-adic integers.

a) State Hensel's Lemma about the factorization of polynomials in the ring $\mathbf{Z}_p[X]$.

b) Show that the field $\mathbb{Q}_p$ contains a primitive $p-1$-root of unity.