Math 530 Comprehensive Exam. August, 2006.
Do any four of the five problems.

Problem 1

Let $\mathcal{O}_K$ be the ring of integers of a number field K.

a) Let $\mathcal{P}$ be a non-zero prime ideal of $\mathcal{O}_K$. Let $r \geq 0$ be an integer. Prove that $\mathcal{O}_K/\mathcal{P}$ and $\mathcal{P}^r/\mathcal{P}^{r+1}$ are isomorphic (as abelian groups, say).

b) Let p be a rational prime which decomposes as

$$p\mathcal{O}_K = \mathcal{P}_1^{e_1} \dots \mathcal{P}_g^{e_g},$$

where the $\mathcal{P}_i$ are distinct prime ideals of $\mathcal{O}_K$. For each i, let $f_i = (\mathcal{O}_K/\mathcal{P}_i : \mathbb{Z}/p\mathbb{Z})$. Prove that $(K : \mathbb{Q}) = \sum_{i=1}^{g} e_i f_i$.

Problem 2

Let $K = \mathbb{Q}(\sqrt{5}, \sqrt{11})$, and let $\mathcal{O}_K$ be its ring of integers.

(a) Let $\mathcal{P}$ be any prime ideal of $\mathcal{O}_K$ which lies over the rational prime 7. Find the decomposition and inertia fields associated to $\mathcal{P}$.

(b) Suppose that $p \in \mathbb{Z}$ is a rational prime. Show that $p\mathcal{O}_K$ is not a prime ideal of $\mathcal{O}_K$.

Problem 3

Let $\Phi_{12}(x) \in \mathbb{Z}[x]$ be the 12th cyclotomic polynomial.

a) Find $\Phi_{12}(x)$ explicitly.

b) Describe the factorization of $\Phi_{12}(x)$ in $\mathbb{Q}_{13}[x]$ ($\mathbb{Q}_{13}$ is the field of 13-adic numbers). Give the number of irreducible factors and their degrees.

Problem 4

a) State Dirichlet's Unit Theorem for the unit group U_K of an algebraic number field K.

b) Let $K = \mathbb{Q}(\zeta_p)$, where p is an odd prime and ζ_p is a primitive p-th root of unity. Prove that $(1 - \zeta_p^r)/(1 - \zeta_p^s)$ is a unit of $\mathcal{O}_K$ if $\gcd(p, rs) = 1$.

c) Now let $K = \mathbb{Q}(\zeta_5)$. Give an explicit set of multiplicatively independent generators for a subgroup of finite index in U_K.

Problem 5

Let K be a number field and let A and B be two non-zero fractional ideals of K.

a) Define what it means for A to be equivalent to B (the equivalence is the standard equivalence used to define the ideal class group).

b) In this part, you may assume the following fact:

> *There exists a positive constant C_K such that every integral ideal of K is equivalent to an ideal whose norm is $\leq C_K$.*

Assuming this, prove that the ideal class group of K is finite.