525 Comp Exam
May 2018

1. Let $X \subseteq \mathbb{R}^2$ be the subset
$$X := S^1 \cup (\mathbb{R} \times \{0\}) \times (\{0\} \times \mathbb{R}),$$
the union of the unit circle, x-axis, and y-axis.

   (a) Describe the fundamental group of X in terms of generators and relations.

   (b) Classify all 2-fold covering maps over X (not necessarily connected).

2. (25 points) Let X be the space obtained from an annulus $\{\, p \in \mathbb{R}^2 \mid 1 \leq |p| \leq 2 \,\}$ by identifying each point (x, y) on the inner circle of radius 1 with the point $(2x, -2y)$ on the outer circle of radius 2.

   (a) Describe a CW-structure for X.

   (b) Compute $\pi_1(X)$ (at any choice of basepoint you wish).

   (c) Compute $H_*(X)$.

3. (25 points)

   Let T be the two dimensional torus.

   Compute the relative homology groups $H_n(T, A)$ for all $n \geq 0$, where A is any finite set of points in T, with $|A| = k$, $k \geq 0$.

4. (25 points) In the following, $p\colon X \to S^1$ and $q\colon Y \to S^1$ will be covering maps, and $s_0 = 1 \in S^1 \subset \mathbb{C}$ is a basepoint for the unit circle. Let $M =$ the set of continuous maps $f\colon X \to Y$ such that $q \circ f = p$.

   (a) Suppose $X = \mathbb{R}$ and $p(t) = e^{2\pi i t}$. Describe and prove bijections $M \xrightarrow{\sim} q^{-1}(s_0)$. (We do not assume that Y is path connected here.)

   (b) Suppose $X = S^1$ and $p(z) = z^2$. Give an example of a covering map $q\colon Y \to S^1$ such that M is empty.