1. Let $X \subseteq \mathbb{R}^2$ be the subset

$$X := S^1 \cup (\mathbb{R} \times \{0\}) \times (\{0\} \times \mathbb{R}),$$

the union of the unit circle, x-axis, and y-axis.

(a) Describe the fundamental group of $X$ in terms of generators and relations.

(b) Classify all 2-fold covering maps over $X$ (not necessarily connected).

2. (25 points) Let $X$ be the space obtained from an annulus $\{ p \in \mathbb{R}^2 \mid 1 \leq |p| \leq 2 \}$ by identifying each point $(x, y)$ on the inner circle of radius 1 with the point $(2x, -2y)$ on the outer circle of radius 2.

(a) Describe a CW-structure for $X$.

(b) Compute $\pi_1(X)$ (at any choice of basepoint you wish).

(c) Compute $H_*(X)$.

3. (25 points)

Let $T$ be the two dimensional torus.

Compute the relative homology groups $H_n(T, A)$ for all $n \geq 0$, where $A$ is any finite set of points in $T$, with $|A| = k, k \geq 0$.

4. (25 points) In the following, $p : X \to S^1$ and $q : Y \to S^1$ will be covering maps, and $s_0 = 1 \in S^1 \subset \mathbb{C}$ is a basepoint for the unit circle. Let $M = \text{the set of continuous maps } f : X \to Y \text{ such that } q \circ f = p$.

(a) Suppose $X = \mathbb{R}$ and $p(t) = e^{2\pi it}$. Describe and prove bijections $M \xrightarrow{\sim} q^{-1}(s_0)$. (We do not assume that $Y$ is path connected here.)

(b) Suppose $X = S^1$ and $p(z) = z^2$. Give an example of a covering map $q : Y \to S^1$ such that $M$ is empty.