525 Comp Exam: May 2017

Problem 1. Let $G = \langle a, b \rangle$ be the free group of rank 2. Use covering spaces to prove the commutator subgroup $[G, G] \leq G$ is not finitely generated.

Problem 2. Let $T = \mathbb{R}^3/\mathbb{Z}^3$ be the 3-dimensional torus, let $B \subset T$ be an embedded 3-dimensional closed ball, and let $Y = T \setminus \operatorname{int}(B)$.

(a) Compute $H_*(T)$.

(b) Compute $H_*(X)$ for $X = (Y \coprod Y)/\sim$ where the quotient is given by gluing the two boundaries $\partial Y = S^2$ together via a homeomorphism.

Problem 3. Using only the Eilenberg–Steenrod axioms:

(a) For all $n \geq 1$, prove $H_*(D^n, S^{n-1}) \cong H_*(S^n, p)$ where $p \in S^n$ is a point.

(b) Using part (a), prove $H_n(S^n) \cong \mathbb{Z}$ for all $n \geq 1$.