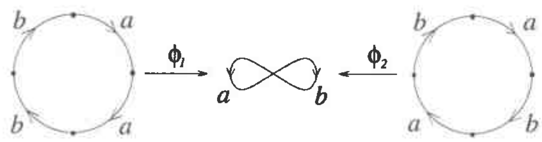
Math 525, Comprehensive Exam, May 2015

1. Let X be the CW complex with one 0–cell, two 1–cells, oriented and labeled a and b, and two 2–cells, attached to the 1–skeleton X^1 by the maps $\phi_1, \phi_2 \colon \partial D^2 \to X^{(1)}$, where ϕ_1 traces the loop a^2b^2 and ϕ_2 traces the loop $abab$.



a. Write down a presentation for $\pi_1(X)$.

b. Calculate the singular homology groups $H_n(X)$ for all $n \geq 0$.

c. Does there exist $1 < d < \infty$ such that X has a connected covering space with d sheets which is normal? Does it have such a covering space which is non-normal? Explain your answer in both cases.

2. Let $n \geq 1$ be an integer, and A be an $n \times n$ matrix with *positive entries*. Prove that A has a positive eigenvalue $\lambda > 0$ with an eigenvector $\mathbf{v}$ having all positive entries. *Hint.* The linear map defined by A restricts to a map in the positive orthant $A \colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ and $A\mathbf{v} \neq \mathbf{0}$ for every $\mathbf{v} \in \mathbb{R}^n_{\geq 0} \setminus \{\mathbf{0}\}$.

3. Consider the standard embedding of the 3–sphere S^3 in $\mathbb{R}^4$:

$$S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\},$$

and let

$$W = \{(x, y, 0, 0) \mid x, y \in \mathbb{R}\} \cup \{(0, 0, z, w) \mid z, w \in \mathbb{R}\}.$$

Setting $L = W \cap S^3$, calculate the relative homology groups $H_n(S^3, L)$ for all $n \geq 0$.