525 Comp Exam.

Wednesday, May 16, 2012.

1. Let X be a space. The *suspension* on X is the quotient space

$$SX = (X \times I)/(\sim)$$

where we make the following identifications:

- all points of the form $(x, 0)$ with $x \in X$ are identified with a single point p_0,
- all points of the form $(x, 1)$ with $x \in X$ are identified with a single point p_1.

Show that there are isomorphisms $H_{n+1}(SX) \approx H_n(X)$ for all non-empty spaces X, and all $n \neq 0$. The *only* facts about homology you should use are the Eilenberg-Steenrod axioms (dimension, sum, exactness, homotopy, excision).

2. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. Let X be the space obtained by attaching a Möbius band M to S^2, so that the boundary circle of M is identified with the equator of S^2.

   (a) Compute the fundamental group of X. (Use any basepoint you wish.)

   (b) Compute the singular homology groups of X.

   ("Compute" can mean "describe in terms of generators and relations".)

3. Let $X = S^1 \vee S^1$, the one-point union of two circles. Classify all two-fold covers of X, up to isomorphism of covers, and for each isomorphism class determine the group of deck transformations. (For the purposes of this problem, covering spaces are not required to be connected.)

4. Let $S^1 \subset \mathbb{R}^2$ be the unit circle, with basepoint s_0. Give an example of:

   (a) a topological space X with basepoint x_0, together with

   (b) two basepoint preserving maps $f_0, f_1 \colon S^1 \to X$, such that

   (c) f_0 and f_1 are related by a homotopy $S^1 \times I \to X$ (non-basepoint preserving), but

   (d) f_0 and f_1 do not represent the same element of $\pi_1(X; x_0)$.

   Justify your answer.