Comprehensive Exam in Topology
University of Illinois, January 2008

1.

Let $I = [0,1]$. Let X be a space and let p and q be two points of X.

a) Give an example of a *connected* space X and points p and q such that

$$\pi_1(X;p) \ncong \pi_1(X,q).$$

b) Show on the other hand that if there is a path $\gamma : I \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$, then there is an isomorphism

$$c_\gamma : \pi_1(X;p) \;\cong\; \pi_1(X;q).$$

c) Show that if X is path-connected and $\pi_1(X;p)$ is an it abelian group, then the isomorphism in **b)** does not depend on the choice of γ: if γ' is another path is X with $\gamma'(0) = p$ and $\gamma'(1) = q$, then

$$c_{\gamma'} = c_\gamma : \pi_1(X;p) \longrightarrow \pi_1(X;q).$$

2. Show that if f is a continuous map from S^2 to S^2 such that $f(-x) = f(x)$ for all x, then $H_2(f) = 0$.

3.

Let X be a space and $U \subseteq X$ open such that $U \cong \mathbb{R}^k$. Show that for $x \in U$,

$$H_n(X, X - \{x\}) \cong \tilde{H}_n(S^k)$$

for all n.

4.

a) Let $f : (X,A) \longrightarrow (Y,B)$ be a map such that both $f : X \longrightarrow Y$ and the restriction $f : A \longrightarrow B$ are homotopy equivalences. Show that $f_* : H_n(X,A) \longrightarrow H_n(Y,B)$ is an isomorphism for all n.

b) Let $I = [0,1]$, $S^0 = \{0,1\}$, and $B = [0,\frac{1}{2})\,(\frac{1}{2},1]$. The identity map induces a map of pairs $(I,S^0) \longrightarrow (I,B)$ which satisfies the conditions of a). Prove that the induced map $H_1(I/S^0) \longrightarrow H_1(I/B)$ is not an isomorphism.

5. Let S^2, D_1 and D_2 be the subspaces of $\mathbb{R}^3$ homeomorphic to a two sphere and two lines determined by

$$S^2 = \{(x_1,x_2,x_3) \mid \sum_{i=1}^{3} x_i^2 = 1\}$$
$$D_1 = \{(x_1,x_2,x_3) \mid -1 \le x_1 \le 1, x_2 = x_3 = 0\}$$
$$D_2 = \{(x_1,x_2,x_3) \mid -1 \le x_2 \le 1, x_1 = x_3 = 0\}$$

Compute the homology of the subspace $X = S^2 \cup D_1 \cup D_2$.