COMPREHENSIVE EXAM IN TOPOLOGY

MAY 2007

1. For $n \geq 0$ let S^n be the n-sphere

$$S^n = \{x \in \mathbb{R}^{n+1} | x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

A *non-vanishing vector field* on S^n is a continuous function

$$v : S^n \to S^n,$$

such that, for all $p \in S^n$, $v(p) \cdot p = 0$ (dot product in $\mathbb{R}^{n+1}$). The *antipodal map* of S^n is the map

$$a(p) = -p$$

(scalar multiplication by -1 in $\mathbb{R}^{n+1}$).

(1) Show that if S^n admits a non-vanishing vector field, then a is homotopic to the identity map.
(2) For which n does S^n have a nowhere vanishing vector field? Justify your answer.

2. If $f : S^1 \to S^1$ is a map, then the *mapping torus* of f is the space

$$T_f = \frac{I \times I}{(0, y) \sim (1, y); (x, 0) \sim (f(x), 1)}.$$

Suppose that f is the map $f(z) = z^n$. Calculate $H_*(T_f; \mathbb{Z})$.

3. Let K be the Klein bottle. Calculate $\pi_1 K$.

4. Let

$$0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$$

be a short exact sequence of chain complexes. Construct the going-around map

$$\delta : H_n(C) \to H_{n-1}(A),$$

and show that

$$H_n(B) \xrightarrow{H_n g} H_n(C) \xrightarrow{\delta} H_{n-1}(A)$$

is exact.